MODULI OF PLANE QUARTICS, GÖPEL INVARIANTS AND BORCHERDS PRODUCTS

SHIGEYUKI KONDŌ

ABSTRACT. It is known that the moduli space of plane quartic curves is birational to an arithmetic quotient of a 6-dimensional complex ball ([Kon1]). In this paper, we shall show that there exists a 15-dimensional space of meromorphic automorphic forms on the complex ball which gives a birational embedding of the moduli space of plane quartics with level 2 structure into $\mathbb{P}^{14}$. This map coincides with the one given by Coble [C] by using Göpel invariants.

CONTENTS

1. INTRODUCTION

In [Kon1], the author proved that the moduli space of smooth plane quartics is isomorphic to an arithmetic quotient $(\mathcal{B} \setminus \mathcal{H})/\Gamma$ where $\mathcal{B}$ is a 6-dimensional complex ball, $\mathcal{H}$ is the union of hypersurfaces in $\mathcal{B}$ and Γ is an arithmetic subgroup of $\mathrm{Aut}(\mathcal{B})$. Moreover $\mathcal{H}$ decomposes into two types: $\mathcal{H} = \mathcal{H}_n \cup \mathcal{H}_h$ where a generic point in $\mathcal{H}_n$ (resp. $\mathcal{H}_h$) corresponds to a plane quartic with a node (resp. a hyperelliptic curve of genus 3). The above isomorphism is defined as follows. Let C be a smooth plane quartic. Then taking the 4-cyclic cover of $\mathbb{P}^2$ branched along C, we have a $K3$ surface X and an automorphism σ of X of order 4. By the theory of periods of $K3$ surfaces, we can see that the period domain of the pairs (X, σ) is $\mathcal{B} \setminus \mathcal{H}$.

There exists a subgroup $\tilde{\Gamma}$ of Γ such that $\Gamma/\tilde{\Gamma} \cong \mathbb{Z}/2\mathbb{Z} \cdot \mathrm{Sp}(6, \mathbb{F}_2)$ and the quotient $(\mathcal{B} \setminus \mathcal{H})/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z})$ is $\mathrm{Sp}(6, \mathbb{F}_2)$-equivariantly isomorphic to the moduli space of smooth plane quartics with level 2-structure (Proposition 5.8). On the other hand, a smooth plane quartic C is naturally corresponding to a del Pezzo surface S of degree two by taking the double cover of $\mathbb{P}^2$ branched along C. Note that S is nothing but the quotient of the above X by the involution σ^2. Conversely the anti canonical model

of S is the double cover of $\mathbb{P}^2$ branched along a smooth quartic. Thus the moduli space of smooth plane quartics is isomorphic to the moduli of del Pezzo surface of degree two. A del Pezzo surface S of degree two is obtained by blowing ups at seven points on $\mathbb{P}^2$ in general position. We call the contraction of $S \to \mathbb{P}^2$ a marking. Here we consider an order of 7 points and 7 exceptional curves. It is known that there are $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ markings. The group of changing markings is isomorphic to $\mathrm{Sp}(6, \mathbb{F}_2) \cong W(E_7)/\{\pm 1\}$ where $W(E_7)$ is the Weyl group of type E_7. This implies that the moduli space of smooth plane quartics with level 2-structure is $\mathrm{Sp}(6, \mathbb{F}_2)$-equivariantly isomorphic to the moduli of ordered 7 points on $\mathbb{P}^2$ in general position. Let P_2^7 be the moduli space of semi-stable ordered 7 points on $\mathbb{P}^2$. Thus we have a birational, $\mathrm{Sp}(6, \mathbb{F}_2)$-equivariant isomorphism between P_2^7 and $(\mathcal{B} \setminus \mathcal{H})/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z})$ (Proposition 7.10).

In [C], by using Göpel invariants, Coble showed that there exists a $\mathrm{Sp}(6, \mathbb{F}_2)$-equivariant birational embedding of P_2^7 into $\mathbb{P}^{14}$ whose image satisfies 63 cubic relations.

In this paper, by using the theory of automorphic forms due to Borcherds [B1], [B2], Freitag [F], we shall show that there exists a 15-dimensional space of *meromorphic* automorphic forms on $\mathcal{B}$ which gives a $\mathrm{Sp}(6, \mathbb{F}_2)$-equivariant birational embedding of $\mathcal{B}/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z})$ into $\mathbb{P}^{14}$ (Theorem 11.1). This map coincides with the one given by Coble [C].

The plan of this paper is as follows. In §2 we fix the notation of lattices. In §3,4, we recall classical results on the moduli of 7 points on $\mathbb{P}^1$ and Coble's work [C], [DO]. In §5,6,7 we study periods of the pairs (X, σ). The sections 8 and 9 devote to construct a linear system of automorphic forms of dimension 15 (Theorem 9.2). To determine the divisor of a member of this linear system, we need a very special automophic form with known zeros and poles. In §10 we show the existence of such an automorphic form (Theorem 10.8, Corollary 10.9, Theorem 10.10, Corollary 10.13). Finally in §11, we shall give the main theorem (Theorem 11.1) and its proof.

We mention the related works. Main idea in this paper follows from the paper Allcock-Freitag [AF] in which they considered the case of del Pezzo surfaces of degree 3, that is, cubic surfaces. They gave a $W(E_6)$-equivariant embedding of the moduli of marked cubic surfaces into $\mathbb{P}^9$ by using automorphic forms. In this case, roughly speaking, automorphic forms correspond to Cayley's cross ratios of cubic surfaces. In case of hyperelliptic curves of genus 3 with level 2-structure or equivalently ordered 8 points on $\mathbb{P}^1$, the author [Kon3] gave a S_8-equivariant embedding of the moduli space into $\mathbb{P}^{13}$ by using Borcherds theory. In this case, automorphic forms correspond to cross ratios of ordered 8 points on $\mathbb{P}^1$. Also the author [Kon2] gave an $O(10, \mathbb{F}_2)^+$-equivariant embedding of the moduli space of Enriques surfaces with "level 2-structure" into $\mathbb{P}^{185}$. A mistake in [Kon2] was corrected in [FS]. The author does not know the geometric meaning of automorphic forms in case of Enriques surfaces. All of these cases used holomorphic automorphic forms. On the other hand, in this paper, the author uses meromorphic automorphic forms.

2. PRELIMINARIES

A *lattice* $(L, \langle, \rangle)$ is a pair of a free $\mathbb{Z}$-module L of rank r and a non-degenerate symmetric bilinear form $\langle, \rangle$ on L. For simplicity we omit $\langle, \rangle$ if there are no confusion. For $x \in L$, we call $x^2 = \langle x, x \rangle$ the *norm* of x. For a lattice $(L, \langle, \rangle)$ and an integer m, we denote by $L(m)$ the lattice $(L, m\langle, \rangle)$. We

denote by U the lattice $(\mathbb{Z}^{\oplus 2}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ and by A_m, D_n or E_k the even *negative* definite lattice defined by the Cartan matrix of type A_m, D_n or E_k respectively. For an integer m, we denote by $\langle m \rangle$ the lattice of rank 1 generated by a vector with norm m. We denote by $L \oplus M$ the orthogonal direct sum of lattices L and M. We also denote by $L^{\oplus k}$ the orthogonal direct sum of k copies of L.

Let L be an even lattice and let $L^* = \mathrm{Hom}(L, \mathbb{Z})$. We denote by A_L the quotient L^*/L and define a map

$$q_L : A_L \to \mathbb{Q}/2\mathbb{Z}, \; b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$$

by $q_L(x + L) = \langle x, x \rangle \bmod 2\mathbb{Z}$ and $b_L(x + L, y + L) = \langle x, y \rangle \bmod \mathbb{Z}$. We call q_L, b_L the *discriminant quadratic form, discriminant bilinear form*, respectively.

Let $\mathrm{O}(L)$ be the orthogonal group of L, that is, the group of isomorphisms of L preserving the bilinear form. Similarly $\mathrm{O}(q_L)$ denotes the group of isomorphisms of A_L preserving q_L. There is a natural map

$$\mathrm{O}(L) \to \mathrm{O}(q_L)$$

whose kernel is denoted by $\tilde{\mathrm{O}}(L)$. For more details we refer the reader to [N1].

3. PLANE QUARTICS AND 7 POINTS ON $\mathbb{P}^2$

Let C be a smooth plane quartic curve in $\mathbb{P}^2$. Then C is a smooth curve of genus three. Conversely the canonical model of a general smooth curve of genus three is a plane quartic. An algebraic surface S is called a *del Pezzo surface of degree* d if the anti canonical class $-K_S$ is ample and $(-K_S)^2 = d$. A smooth plane quartic C is naturally corresponding to a del Pezzo surface S of degree two by taking the double cover of $\mathbb{P}^2$ branched along C. Conversely the anti canonical model of S is the double cover of $\mathbb{P}^2$ branched along a smooth quartic. Thus the moduli space of smooth plane quartics is isomorphic to the moduli of del Pezzo surface of degree two. It is well known that there are 28 bitangent lines to C. On S, bitangent lines split into 56 (-1)-curves.

A del Pezzo surface S of degree two is obtained by blowing ups at seven points on $\mathbb{P}^2$ in general position. Here seven points $p_1, ..., p_7$ in $\mathbb{P}^2$ are *in general position* if no two points coincide, no three points lie on a line and no six points lie on a conic. Let $e_1, ..., e_7$ be exceptional curves and e_0 the total transform of the line. The Picard lattice of S is isomorphic to $\langle 1 \rangle \oplus \langle -1 \rangle^{\oplus 7}$ where $\langle 1 \rangle$ is generated by e_0 and $\langle -1 \rangle^{\oplus 7}$ is generated by $e_1, ..., e_7$. The orthogonal complement of the anti-canonical class $3e_0 - e_1 - \cdots - e_7$ is isomorphic to the root lattice E_7. There are 56 (-1)-curves, that is, 7 exceptional curves, proper transforms of 21 lines through two points from $p_1, ..., p_7$, proper transforms of 21 conics throuh 5 points from $p_1, ..., p_7$, proper transforms of 7 cubics through 7 points and having a node at one of 7 points:

(3.1)
$$\begin{cases} e_i \; (1 \leq i \leq 7), \\ e_0 - e_i - e_j \; (1 \leq i < j \leq 7), \\ 2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_i + e_j \; (1 \leq i < j \leq 7), \\ 3e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_i \; (1 \leq i \leq 7). \end{cases}$$

On the other hand, $p_1, ..., p_7$ are in general position iff all of the 63 divisors

$$(3.2) \qquad \begin{cases} e_i - e_j \ (1 \le i < j \le 7), \\ e_0 - e_i - e_j - e_k \ (1 \le i < j < k \le 7), \\ 2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_i \ (1 \le i \le 7) \end{cases}$$

on S are not effective. We call (3.2) *the discriminant conditions* of 7 points in $\mathbb{P}^2$. Note that 63 divisors in the discriminant condition (3.2) can be considered as a set of positive roots of E_7. There are $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ sets of disjoint seven (-1)-curves on S, in other words, there are $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ contractions of S to $\mathbb{P}^2$. We call a contraction $S \to \mathbb{P}^2$ a *marking* of S. The group of changing markings is isomorphic to $\mathrm{Sp}(6, \mathbb{F}_2)$. Let $W(E_7)$ be the Weyl group of E_7. Then there is an exact sequence

$$(3.3) \qquad\qquad 1 \to \{\pm 1\} \to W(E_7) \to \mathrm{Sp}(6, \mathbb{F}_2) \to 1$$

where $\{\pm 1\}$ is generated by the covering transformation of $S \to \mathbb{P}^2$. A marking of S naturally corresponds to a level 2 structure of a smooth curve C of genus 3. Thus the moduli space of del Pezzo surfaces of degree 2 with a marking is isomorphic to the moduli of smooth plane quartics with a level 2 structure. This isomorphism is $\mathrm{Sp}(6, \mathbb{F}_2)$-equivariant. Let P_2^7 be the moduli space of semi-stable ordered 7 points on $\mathbb{P}^2$. Here a set of ordered 7 points $\{p_1, ..., p_7\}$ is *semi-stable* (=*stable*) iff at most two points coincide and at most 4 points are collinear (e.g. [DO], p.120). The discriminat condition (3.2) gives 63 subvarieties in P_2^7 whose complement parametrizes smooth plane quartics.

For more details, we refer the reader to [DO].

4. GÖPEL INVARIANTS

In the following we shall recall Coble's work [C] (also see [DO]) on Göpel invariants. For simplicity we denote by $\alpha(i, j)$, $\alpha(i, j, k)$, $\alpha(i)$ the positive roots $e_i - e_j$, $e_0 - e_i - e_j - e_k$, $2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_i$, respectively.

The structure of lattice on E_7 induces a structure of quadratic form on $E_7/2E_7$ over $\mathbb{F}_2$ and of dimension 7. It has a radical $\sum_{i=0}^{7} e_i \bmod 2E_7$ and modulo radical it induces a symplectic space $\mathbb{F}_2^6$ of dimension 6.

A maximal totally isotropic subspace of $\mathbb{F}_2^6$ is called *Göpel subspace*. There are 135 Göpel subspaces. Each non-zero element in $\mathbb{F}_2^6$ corresponds to a positive root in E_7 and 7 non-zero elements in a Göpel subspace gives a set of mutually orthogonal 7 positive roots ([DO], Chap. IX, Lemma 8). We call a set of mutually orthogonal 7 positive roots a *Göpel subset*. There are 135 Göpel subsets ([C], §28):

$$(4.1) \qquad \begin{cases} 30 \text{ of type } \{\alpha(1, 2, 3), \alpha(1, 4, 5), \alpha(2, 4, 6), \alpha(3, 5, 6), \alpha(1, 6, 7), \alpha(2, 5, 7), \alpha(3, 4, 7)\}, \\ 105 \text{ of type } \{\alpha(1), \alpha(1, 2, 3), \alpha(1, 4, 5), \alpha(1, 6, 7), \alpha(2, 3), \alpha(4, 5), \alpha(6, 7)\}. \end{cases}$$

For each Göpel subset (or corresponding Göpel subspace) M, we define a function G_M on $(\mathbb{C}^3)^7$ called *Göpel invariant* as follows. First consider a Göpel subset M given by

$$(4.2) \qquad \{\alpha(1, 2, 3), \alpha(1, 4, 5), \alpha(2, 4, 6), \alpha(3, 5, 6), \alpha(1, 6, 7), \alpha(2, 5, 7), \alpha(3, 4, 7)\}.$$

Let p_i, p_j, p_k be three points in $\mathbb{P}^2$ and let v_i, v_j, v_k the column vectors in $\mathbb{C}^3$ corresponding to p_i, p_j, p_k respectively. We denote by (ijk) the discriminant of the 3×3 matrix $(v_i v_j v_k)$. Then the Göpel

invariant G_M is defined by

(4.3) $$G_M = (123)(145)(246)(356)(167)(257)(347).$$

Obviously G_M vanishes along 7 divisors corresponding to 7 roots in M. Next consider the case M is given by

(4.4) $$\{\alpha(1), \alpha(1,2,3), \alpha(1,4,5), \alpha(1,6,7), \alpha(2,3), \alpha(4,5), \alpha(6,7)\}.$$

Then there are exactly three Göpel subsets M, M_1, M_2 containing $\alpha(1,2,3), \alpha(1,4,5), \alpha(1,6,7)$:

(4.5) $$M_1 = \{\alpha(1,2,3), \alpha(1,4,5), \alpha(2,4,6), \alpha(3,5,6), \alpha(1,6,7), \alpha(2,5,7), \alpha(3,4,7)\}.$$

(4.6) $$M_2 = \{\alpha(1,2,3), \alpha(1,4,5), \alpha(2,5,6), \alpha(3,5,7), \alpha(1,6,7), \alpha(2,4,7), \alpha(3,4,6)\}.$$

Then G_M is defined by

$$G_M = \pm(G_{M_1} - G_{M_2}) = (123)(145)(167)\{(246)(356)(257)(347) - (256)(357)(247)(346)\}$$

up to sign $\pm$. Then we can easily see that the second factor of G_M vanishes along 4 subvarieties corresponding to $\alpha(1), \alpha(2,3), \alpha(4,5), \alpha(6,7)$. For example, if we fix 5 points p_2, p_3, p_4, p_5, p_6 and consider p_7 as a parameter, then the second factor of G_M is a conic through these 5 points.

4.1. **Proposition.** (Coble [C], [DO], Chap. IX) (1) *Assume that M is one of 30 types, for example, M is given by (4.2). Then G_M vanishes exactly along 7 divisors corresponding to 7 roots in M with multiplicity one. Moreover G_M vanishes along 21 subvarieties corresponding to 21 roots $\alpha(i,j)$ with multiplicity one.*

(2) *Assume that M is one of 105 types, for example, M is given by (4.4). Then G_M vanishes along 7 subvarieties corresponding to 7 roots in M plus 21 subvarieties corresponding to 21 roots $\alpha(i,j)$.*

(3) *Let A be a 2-dimensional totally isotropic subspace. Then there are exactly three Göpel subspaces M_1, M_2, M_3 containing A. Moreover G_{M_i} $(i = 1, 2, 3)$ satisfy a linear relation*

$$G_{M_1} \pm G_{M_2} \pm G_{M_3} = 0$$

where the sign $\pm$ is taken for a suitable one. The 135 Göpel invariants generate a 15-dimensional space on which $Sp(6, \mathbb{F}_2)$ acts linearly via its action on Göpel subspaces. This 15-dimensional representation of $Sp(6, \mathbb{F}_2)$ is irreducible.

4.2. **Theorem.** (Coble [C], Chap. IV, [DO], Chap. IX, Theorem 5) *The 15-dimensional linear system defines a $Sp(6, \mathbb{F}_2)$-equivariant rational map ψ from P_2^7 to $\mathbb{P}^{14}$ which is birational onto its image. The image satisfies 63 cubic relations corresponding to 63 positive roots.*

4.3. **Lemma.** *Each G_M vanishes with multiplicity 1 along exactly 28 subvarieties among 36 subvarieties defined by*:

(4.7) $$\begin{cases} e_0 - e_i - e_j - e_k - e_l, \ 1 \le i < j < k < l \le 7, \\ 2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7, \end{cases}$$

i.e. four points are collinear or seven points lie on a conic.

Proof. Since $\mathrm{Sp}(6, \mathbb{F}_2)$ acts transitively on the set of Göpel invariants, it is enough to see the assertion for a Göpel invariant. Consider the case

$$G_M = (123)(145)(246)(356)(167)(257)(347).$$

Then G_M does not vanish identically along 8 subvarieties defined by

$$2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7, e_0 - e_i - e_j - e_k - e_l$$

where

$$(i, j, k, l) = (1, 2, 4, 7), (1, 2, 5, 6), (1, 3, 4, 6), (1, 3, 5, 7), (2, 3, 4, 5), (2, 3, 6, 7), (4, 5, 6, 7).$$

And G_M vanishes along another 28 subvarieties with multiplicity 1. □

Let $\pi : \hat{P}_2^7 \to P_2^7$ be the blow ups of P_2^7 along 21 subvarieties corresponding to 21 roots $e_i - e_j$ ($1 \leq i < j \leq 7$) ([DO], Chap. IV). Thus $\hat{P}_2^7$ has 63 divisors corresponding to 63 discriminant conditions. For a positive root α, we denote by D_α the corresponding divisor. For Göpel subsets M_1, M_2, we denote by $\hat{G}_{M_1}/\hat{G}_{M_2}$ the pull back of G_{M_1}/G_{M_2} by π. Then

4.4. **Lemma.** *The divisor of $\hat{G}_{M_1}/\hat{G}_{M_2}$ is given by*

$$\sum_{\alpha \in M_1} D_\alpha - \sum_{\beta \in M_2} D_\beta.$$

Proof. We remark that both G_{M_1} and G_{M_2} vanish along 21 subvarieties corresponding to 21 roots $\alpha(i, j)$ with multiplicity one. Hence the assertion follows from Proposition 4.1. □

Later we shall construct automorphic forms with the same property as Göpel invariants. See Corollaries 10.11, 10.13

5. $K3$ SURFACES ASSOCIATED TO PLANE QUARTIC CURVES

5.1. $K3$ **surfaces and a bounded symmetric domain of type** IV**.** In [Kon1], the author proved that the moduli space of plane quartic curves is birational to an arithmetic quotient of a 6-dimensional complex ball. We recall this fact briefly. Let C be a smooth plane quartic curve and let $f_4(x, y, z) = 0$ be the defining equation of C. Consider the smooth quartic surface

$$X : f_4(x, y, z) + t^4 = 0.$$

Obviously X is a 4 cyclic cover of $\mathbb{P}^2$ branched along C and is a $K3$ surface, that is, the canonical class K_X is trivial and $H^1(X, \mathcal{O}_X) = 0$. If we denote by S the double cover of $\mathbb{P}^2$ branched along C, then the 4 cyclic cover factorizes and X is the double cover of S branched along C. Let σ be an automorphism of X of order 4 which is the covering transformation of $X \to \mathbb{P}^2$ and let $\iota = \sigma^2$. The second cohomology group $H^2(X, \mathbb{Z})$ together with the cup product adimits a structure of lattice. As a lattice, $H^2(X, \mathbb{Z})$ is an even unimodular lattice of signature $(3, 19)$ which is uniquely determined by this property. We denote by L an abstract even unimodular lattice of signature $(3, 19)$. Then $H^2(X, \mathbb{Z}) \cong L$. We define

$$H^2(X, \mathbb{Z})_+ = \{x \in H^2(X, \mathbb{Z}) : \iota^*(x) = x\}, \ H^2(X, \mathbb{Z})_- = \{x \in H^2(X, \mathbb{Z}) : \iota^*(x) = -x\}.$$

Both $H^2(X, \mathbb{Z})_\pm$ are 2-elementary lattices which are isomorphic to $L_\pm$ respectively where

(5.1) $L_+ = A_1(-1) \oplus A_1^{\oplus 7}, \ L_- = U \oplus U(2) \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}.$

We denote by $\tilde{e}_i$ $(0 \le i \le 7)$ the pullback of e_i by the map $X \to S$. Then $\tilde{e}_0$ generates $A_1(-1)$ in (5.1) and $\tilde{e}_1, ..., \tilde{e}_7$ generate $A_1^{\oplus 7}$. Let

$$q_{L_+} : A_{L_+} = L_+^*/L_+ \to \mathbb{Q}/2\mathbb{Z}$$

be the discriminant quadratic form of L_+. Define $\tilde{\kappa} = 3\tilde{e}_0 - \tilde{e}_1 - \cdots - \tilde{e}_7$. Then

5.2. Lemma. (e.g. [Kon1], Lemma 2.1) (i) *A vector x in A_{L_+} with $q_{L_+}(x) = -1/2$ is represented by one of the following 56 vectors*:

$$\tilde{e}_i/2, (\tilde{\kappa} - \tilde{e}_i)/2, 1 \le i \le 7, (\tilde{e}_0 - \tilde{e}_i - \tilde{e}_j)/2, (\tilde{\kappa} - \tilde{e}_0 + \tilde{e}_i + \tilde{e}_j)/2, 1 \le i < j \le 7.$$

(ii) *A vector x in A_{L_+} with $q_{L_+}(x) = 1/2$ is represented by one of the following 72 vectors*:

$$\tilde{e}_0/2, (2\tilde{e}_0 - \sum_i \tilde{e}_i)/2, (\tilde{e}_0 - \tilde{e}_i - \tilde{e}_j - \tilde{e}_k - \tilde{e}_l)/2, (2\tilde{e}_0 - \tilde{e}_i - \tilde{e}_j - \tilde{e}_k)/2, 1 \le i < j < k < l \le 7.$$

(iii) *A vector x in A_{L_+} with $q_{L_+}(x) = 1$ is represented by one of the following 64 vectors*:

$$\tilde{\kappa}/2, (\tilde{e}_i - \tilde{e}_j)/2, 1 \le i < j \le 7, (\tilde{e}_0 - \tilde{e}_i - \tilde{e}_j - \tilde{e}_k)/2, 1 \le i < j < k \le 7,$$

$$(2\tilde{e}_0 - \sum_i \tilde{e}_i + \tilde{e}_j)/2, 1 \le j \le 7.$$

(iv) *A vector x in A_{L_+} with $q_{L_+}(x) = 0$ is represented by one of the following 64 vectors*:

$$0, (\tilde{\kappa} + \tilde{e}_i - \tilde{e}_j)/2, 1 \le i < j \le 7, (\tilde{\kappa} + \tilde{e}_0 - \tilde{e}_i - \tilde{e}_j - \tilde{e}_k)/2, 1 \le i < j < k \le 7,$$

$$(\tilde{\kappa} + 2\tilde{e}_0 - \sum_i \tilde{e}_i + \tilde{e}_j)/2, 1 \le j \le 7.$$

(v) σ^* *acts trivially on vectors in* (iii), (iv) *and acts on vectors x in* (i), (ii) *as $\sigma^*(x) = \tilde{\kappa}/2 - x$.*

5.3. Remark. (1) Note that 63 vectors in Lemma 5.2, (iii), except $\tilde{\kappa}/2$ correspond to 63 discriminant conditions (3.2). Also 72 vectors in (ii) modulo the action of σ^* correspond to 36 conditions (4.7) in Lemma 4.3.

(2) Consider the subspace A'_{L_+} of A_{L_+} on which q_{L_+} takes integral values. Then (A'_{L_+}, q_{L_+}) is a quadratic form of dimension 7 over $\mathbb{F}_2$. It has a radical generated by $\tilde{\kappa}/2$.

If $\omega \in H^0(X, \Omega_X^2)$, then $\iota^*(\omega_X) = -\omega_X$, and hence the period domain of the above $K3$ surfaces is given by

(5.2) $$\mathcal{D} = \{[\omega] \in \mathbb{P}(L_- \otimes \mathbb{C}) : \langle \omega, \omega \rangle = 0, \ \langle \omega, \bar{\omega} \rangle > 0\}$$

which is a bounded symmetric domain of type IV and of dimension 12.

5.4. An isometry of order 4. We shall study the action of σ on $H^2(X, \mathbb{Z})^-$. Recall that

$$D_4 \cong \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1 + x_2 + x_3 + x_4 \equiv 0 \, (\mathrm{mod}\, 2)\}.$$

Here we consider the standard inner product on $\mathbb{Z}^4$ with the *negative* sign. Let ρ_0 be the isometry of D_4 given by

$$\rho_0(x_1, x_2, x_3, x_4) = (x_2, -x_1, x_4, -x_3).$$

Obviously ρ_0 is of order 4 and fixes no non-zero vectors in D_4. Also an easy calculation shows that ρ_0 acts trivially on D_4^*/D_4. Next let e, f (resp. e', f') be a basis of U (resp. $U(2)$). Define the isometry ρ_1 of $U \oplus U(2)$ by

$$\rho_1(e) = -e - e', \ \rho_1(f) = f - f', \ \rho_1(e') = e' + 2e, \ \rho_1(f') = 2f - f'.$$

Obviously ρ_1 is of order 4, fixes no non-zero vectors in $U \oplus U(2)$ and acts trivially on the discriminant group of $U \oplus U(2)$.

Finally let ρ_2 be an isometry of $A_1^{\oplus 2}$ with a basis r_1, r_2 defined by

$$\rho_2(r_1) = -r_2, \ \rho_2(r_2) = r_1.$$

Now we have an isometry $\rho = \rho_1 \oplus \rho_0 \oplus \rho_0 \oplus \rho_2$ of order 4 of $L_- = U \oplus U(2) \oplus D_4 \oplus D_4 \oplus A_1 \oplus A_1$ which fixes no non-zero vectors in L_-. The action of ρ on L_-^*/L_- coincides with the one of σ^* on $H^2(X, \mathbb{Z})_-^*/H^2(X, \mathbb{Z})_-$.

5.5. A complex ball as the period domain of $K3$ surfaces with an automorphism of order 4.
Now we can define the period domain of the pairs (X, σ). Since $H^0(X, \Omega_X^2) \cong \mathbb{C}$, the period ω_X is an eigenvector of σ^*. First note that ρ has no non-zero fixed vectors in $L_- \otimes \mathbb{Q}$ and hence the eigenvalues of ρ are $\pm\sqrt{-1}$. Let

$$V_\pm = \{\omega \in L_- \otimes \mathbb{C} : \rho(\omega) = \pm\sqrt{-1}\omega\}$$

both of which have dimension 7. Then the period domain of the pairs (X, σ) is defined by

$$(5.3) \qquad\qquad\qquad\qquad \mathcal{B} = \mathcal{D} \cap \mathbb{P}(V_+).$$

If $\omega \in V_\pm$, then $\langle \omega, \omega \rangle = 0$. This shows

$$\mathcal{B} = \{[\omega] \in \mathbb{P}(V_+) : \langle \omega, \bar{\omega} \rangle > 0\}$$

and hence $\mathcal{B}$ is a 6-dimensional complex ball.

Let $r \in L_-$ with $r^2 = \langle r, r \rangle = -2$. We define

$$(5.4) \qquad\qquad r^\perp = \{\omega \in \mathcal{D} \ : \ \langle \omega, r \rangle = 0\}, \ H_r = r^\perp \cap \mathcal{B}, \ \mathcal{H} = \bigcup_{r \in L_-, r^2 = -2} H_r$$

5.6. Proposition. *There are two types of (-2)-vectors r in L_- according to $r/2 \in L_-^*$ or $r/2 \notin L_-^*$.*

Proof. See [Kon1], Lemma 3.3. □

According to the above Proposition, we can write

$$\mathcal{H} = \mathcal{H}_n \cup \mathcal{H}_h$$

where $\mathcal{H}_n$ (resp. $\mathcal{H}_h$) is the union of $r^\perp$ with $r/2 \notin L_{L_-}^*$ (resp. $r/2 \in L_{L_-}^*$). We showed that a generic point in $\mathcal{H}_n$ (resp. $\mathcal{H}_h$) corresponds to a plane quartic with a node (resp. a hyperelliptic curve of genus 3) ([Kon1], Theorems 4.3, 5.3). We shall study more details of $\mathcal{H}$ in 7.4.

Also we consider the following arithmetic subgroups

$$(5.5) \qquad\qquad \Gamma = \{\gamma \in \mathrm{O}(L_-) : \gamma \circ \rho = \rho \circ \gamma\}, \ \tilde{\Gamma} = \mathrm{Ker}\{\Gamma \to \mathrm{O}(q_{L_-})\}.$$

Then $\Gamma/\tilde{\Gamma}$ is isomorphic to a split extension of $\mathrm{Sp}(6, \mathbb{F}_2)$ by $\mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ is generated by ρ (e.g. [Kon1], Lemma 2.2).

5.7. **Proposition.** *The action of σ^* on $H^2(X, \mathbb{Z})_-$ is conjugate to ρ.*

Proof. Recall that the action of ρ on L_-^*/L_- coincides with the one of σ^* on $H^2(X, \mathbb{Z})_-^*/H^2(X, \mathbb{Z})_-$. It follows from Nikulin [N1], Proposition 1.6.1 that ρ can be extended to an isometry of L whose action on L_+ isomorphic to the one of σ^* on $H^2(X, \mathbb{Z})_+$. Let $\omega \in \mathcal{B} \setminus \mathcal{H}$. Then the surjectivity of the period map for $K3$ surfaces, there exists a $K3$ surface Y and an isometry $\alpha_Y : H^2(Y, \mathbb{Z}) \to L$ with $\alpha_Y(\omega_Y) = \omega$. Then ρ is represented by an automorphism σ' ([Kon1], Lemma 2.4). It follows from Nikulin [N2], Theorem 4.2.2 that the set of fixed points of $(\sigma')^2$ is a smooth curve C' of genus 3. We can easily see that the set of fixed points of σ' is C', the quotient surface $Y/\langle \sigma' \rangle$ is a projective plane and the branch locus is a smooth plane quartic. Since the moduli space of plane quartics is connected, the assertion follows. $\square$

5.8. **Proposition.** ([Kon1], Theorem 2.5) *The moduli space of smooth plane quartics (resp. plane quartics with a level 2 structure) is isomorphic to the quotient $(\mathcal{B} \setminus \mathcal{H})/\Gamma$ (resp. $(\mathcal{B} \setminus \mathcal{H})/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z}))$.*

6. HERMITIAN FORM AND REFLECTIONS

6.1. **Hermitian form.** We consider L_- as a free $\mathbb{Z}[\sqrt{-1}]$-module Λ by

$$(a + b\sqrt{-1})x = ax + b\rho(x).$$

Let

$$h(x, y) = \sqrt{-1}\langle x, \rho(y) \rangle + \langle x, y \rangle.$$

Then $h(x, y)$ is a hermitian form on $\mathbb{Z}[\sqrt{-1}]$-module Λ. With respect to a $\mathbb{Z}[\sqrt{-1}]$-basis $(1, -1, 0, 0)$, $(0, 1, -1, 0)$ of D_4, the hermitian matrix of $h \mid D_4$ is given by

(6.1)
$$\begin{pmatrix} -2 & 1 - \sqrt{-1} \\ 1 + \sqrt{-1} & -2 \end{pmatrix}.$$

And with respect to a $\mathbb{Z}[\sqrt{-1}]$-basis e, e' of $U \oplus U(2)$, the hermitian matrix of $h \mid U \oplus U(2)$ is given by

(6.2)
$$\begin{pmatrix} 0 & 1 + \sqrt{-1} \\ 1 - \sqrt{-1} & 0 \end{pmatrix}.$$

And with respect to a $\mathbb{Z}[\sqrt{-1}]$-basis r of $A_1 \oplus A_1$, the hermitian matrix of $h \mid A_1 \oplus A_1$ is given by (-2). Thus the hermitian matrix of h is given by

(6.3)
$$\begin{pmatrix} 0 & 1 + \sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\ 1 - \sqrt{-1} & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & -2 & 1 - \sqrt{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 + \sqrt{-1} & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 - \sqrt{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 + \sqrt{-1} & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Let

$$\varphi : \Lambda \to L_-^*$$

be a linear map defined by $\varphi(x) = (x + \rho(x))/2$. Note that $\varphi((1 - \sqrt{-1})x) = \varphi(x - \rho(x)) = x \in L_-$. Hence φ induces an isomorphism

$$\text{(6.4)} \qquad\qquad \Lambda/(1 - \sqrt{-1})\Lambda \simeq L_-^*/L_-.$$

6.2. **Reflections.** For $r \in L_-$ with $\langle r, r \rangle = -2$, we define a *reflection*

$$s_r(x) = x + \langle r, x \rangle r$$

which is contained in $\tilde{O}(L_-) = \text{Ker}(O(L_-) \to O(q_{L_-}))$, but not in $\text{Aut}(\Lambda) \cong \Gamma$. Also for $\xi \in L_-$ with $\xi^2 = -4$ and $\xi/2 \in L_-^*$, we define a reflection s_ξ in $O(L_-)$ by

$$s_\xi(x) = x + \frac{1}{2}\langle x, \xi \rangle \xi.$$

The reflection s_ξ induces a *transvection* of A_{L_-} defined by

$$t_\alpha(x) = x + 2b_{L_-}(x, \alpha)\alpha$$

where $\alpha \in A_{L_-}$ is a non-isotropic vector represented by $\xi/2$.

On the other hand, by considering r as in Λ, we define a *reflection* in Γ by

$$\text{(6.5)} \qquad\qquad R_{r,\epsilon}(x) = x - (1 - \epsilon)\frac{h(r, x)}{h(r, r)}r$$

where $\epsilon \neq 1$ is a 4-th root of unity. We can easily see that $R_{r,-1}$ corresponds to the isometry of L_-

$$x \to x + \langle r, x \rangle r + \langle \rho(r), x \rangle \rho(r)$$

which coincides with $s_r \circ s_{\rho(r)}$. Also $R_{r,\sqrt{-1}}$ corresponds to the isometry of L_-

$$x \to x + \langle r, x \rangle(r - \rho(r))/2 + \langle \rho(r), x \rangle(r + \rho(r))/2$$

which induces a transvection t_α of A_{L_-} where $\alpha \in A_{L_-}$ is a non-isotropic vector $(r + \rho(r))/2 \mod L_-$. We can easily see that

$$R_{r,\sqrt{-1}}^2 = s_{r - \rho(r)} \circ s_{r + \rho(r)}.$$

Moreover a direct calculation shows that $R_{r,\sqrt{-1}}^2 = R_{r,-1}$. We remark that $R_{r,\sqrt{-1}}^2$ acts trivially on A_{L_-}. It is known that the transvections t_α ($\alpha \in A_{L_-}$, $q_{L_-}(\alpha) = 1$) generate $O(q_{L_-})$. Thus we have

6.3. **Proposition.** *The natural map*

$$\Gamma \to O(q_{L_-})$$

is surjective.

7. HEEGNER DIVISORS

In this section we shall study the discriminant locus $\mathcal{H}$.

7.1. **Discriminant quadratic form.** Recall that $L_+ = A_1(-1) \oplus A_1^{\oplus 7}$, $L_- = U \oplus U(2) \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}$.

Let $(A_{L_-}, q_{L_-}) = (L_-^*/L_-, q_{L_-})$ be the discriminant quadratic form of L_-. Then $A_{L_-} \simeq (\mathbb{F}_2)^8$ and

$$q_{L_-} : A_{L_-} \to \mathbb{Q}/2\mathbb{Z}.$$

Since L_+ and L_- are mutually orthogonal complement in the unimodular lattice L, A_{L_+} is canonically isomorphic to A_{L_-}. Define

$$A'_{L_-} = \{x \in A_{L_-} : q_{L_-}(x) \in \mathbb{Z}/2\mathbb{Z}\}.$$

Then the quadratic form $q_{L_-} \mid A'_{L_-}$ over $\mathbb{F}_2$ has a radical $\langle \kappa \rangle$ and modulo radical it defines a symplectic form of dimension 6 over $\mathbb{F}_2$ (see Remark 5.3, (2)).

A_{L_-} consists of the following 256 vectors:

   Type (00): $q_{L_-}(x) = 0, \#x = 1, x = 0$;

   Type (0): $q_{L_-}(x) = 0, \#x = 63$;

   Type (1): $q_{L_-}(x) = 1, \#x = 63$;

   Type (10): $q_{L_-}(x) = 1, \#x = 1. \ x = \kappa$;

   Type (1/2): $q_{L_-}(x) = 1/2, \#x = 56$;

   Type (3/2): $q_{L_-}(x) = 3/2, \#x = 72$.

Since $\mathrm{O}(L_-)$ acts transitively on vectors in A_{L_-} with the same type, the number of $\tilde{\mathrm{O}}(L_-)$-equivalence classes are 1, 63, 63, 1, 56 or 72 according to the type (00), (0), (1), (10), (1/2) or (3/2) respectively.

7.2. **Lemma.** *The quadratic space $E_7/2E_7$ is isomorphic to A'_{L_-}.*

Proof. First note that $q_{L_+} = -q_{L_-}$ because L_- is the orthogonal complement of L_+ in the unimodular lattice L. On the other hand, the Picard lattice of a del Pezzo surface S of degree 2 is isomorphic to $L_+(1/2)$. Moreover $E_7/2E_7$ is isomorphic to $q_{L_-} \mid A'_{L_-}$ by using the description of A_{L_+} in Lemma 5.2. $\qquad\square$

7.3. **Remark.** Under the isomorphism in Lemma 7.2, 63 discriminant conditions (3.2) (resp. 36 conditions (4.7) in Lemma 4.3) correspond to 63 vectors of type (1) (resp. 72 vectors of type (3/2)) modulo the action of ρ). See Remark 5.3.

7.4. **Heegner divisors.** We now introduce Heegner divisors in $\mathcal{D}$ and $\mathcal{B}$. Let $r \in L_-^*$ with $r^2 < 0$. We denote by $r^\perp$ the orthogonal complement in $\mathcal{D}$:

$$r^\perp = \{[\omega] \in \mathcal{D} : \langle r, \omega \rangle = 0\}.$$

Let $\alpha \in A_{L_-}$ and $n \in \mathbb{Q}$, $n < 0$. Then we difine a *Heegner divisor* $\mathcal{D}_{\alpha,n}$ in $\mathcal{D}$ as follows:

$$\mathcal{D}_{\alpha,n} = \bigcup_r r^\perp$$

where r moves over the set of all $r \in L_-^*$ satisfying $r \bmod L_- = \alpha$ and $r^2 = n$.

In the later we consider only special cases, that is, α is of type (1) , (10) and $n = -1$, α is of type (1/2) and $n = -3/2$, and α is of type (3/2) and $n = -1/2$. In these cases, for simplicity, we denote

$\mathcal{D}_{\alpha,n}$ by $\mathcal{D}_\alpha$: if α is of type (1) or type (10), $\mathcal{D}_\alpha = \mathcal{D}_{\alpha,-1}$, if α is of type (1/2), $\mathcal{D}_\alpha = \mathcal{D}_{\alpha,-3/2}$ and if α is of type (3/2), $\mathcal{D}_\alpha = \mathcal{D}_{\alpha,-1/2}$. Moreover for $j = 1, 10, 1/2$ or $3/2$, we define

$$\mathcal{D}_j = \bigcup_{\alpha \text{ is of type } (j)} \mathcal{D}_\alpha.$$

Let $r \in L_-^*$ with $r^2 < 0$. Since $\langle r, \rho(r) \rangle = \langle \rho(r), \rho^2(r) \rangle = -\langle r, \rho(r) \rangle$, $\langle r, \rho(r) \rangle = 0$. If $\omega \in \mathcal{B}$, then $\langle r, \omega \rangle = \sqrt{-1}\langle \rho(r), \omega \rangle$. Thus we have

$$\langle r, \omega \rangle = 0 \iff \langle \rho(r), \omega \rangle = 0 \iff \langle r + \rho(r), \omega \rangle = 0 \iff \langle r - \rho(r), \omega \rangle = 0.$$

Therefore

(7.1) $$H_r = H_{\rho(r)} = H_{r+\rho(r)} = H_{r-\rho(r)}$$

where $H_r = r^\perp \cap \mathcal{B}$.

7.5. **Lemma.** *Let $r \in L_-$ with $r^2 = -2$. If $r/2 \notin L_-^*$, then $(r + \rho(r))/2$ is contained in L_-^* and $(r + \rho(r))/2 \bmod L_-$ is of type (1). If $r/2 \in L_-^*$, then $(r + \rho(r))/2 \bmod L_-$ is of type (10).*

Proof. The assertion follows from [Kon1], Lemma 3.3 and its proof.   ☐

7.6. **Lemma.** *Let $r \in L_-$ with $r^2 = -2$.*
   (1) *The order of the reflection $R_{r,\sqrt{-1}}$ is 4 and $R_{r,\sqrt{-1}}^2$ is contained in $\tilde{\Gamma}$.*
   (2) *If $r/2 \notin L_-^*$, then the reflection $R_{r,\sqrt{-1}}$ is not contained in $\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z}$.*
   (3) *If $r/2 \in L_-^*$, then $R_{r,\sqrt{-1}}$ is contained in $\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z}$.*

Proof. A direct calculation shows that $R_{r,\sqrt{-1}}$ is of order 4. As mentioned in 6.2, $R_{r,\sqrt{-1}}^2$ acts trivially on A_{L_-}. On the other hand, $R_{r,\sqrt{-1}}$ induces a transvection t_α on A_{L_-} where $\alpha = (r + \rho(r))/2 \bmod L_-$ (see 6.2), and hence it acts non trivially on A_{L_-}. If $r/2 \notin L_-^*$ (resp. $r/2 \in L_-^*$), then α is of type (1) (resp. of type (10)) (Lemma 7.5). Moreover if α is of type (10), then it acts on A_{L_-} as ρ. Hence the assertion follows.   ☐

For $\alpha \in A_{L_-}$, define a *Heegner divisor* in $\mathcal{B}$ by

$$\mathcal{B}_\alpha = \mathcal{B} \cap \mathcal{D}_\alpha$$

and denote its image on $\mathcal{B}/\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z}$ by $\mathcal{H}_\alpha$. It follows from the equation 7.1, Proposition 5.6, Lemma 7.5 that

$$\mathcal{H} = \bigcup_\alpha \mathcal{B}_\alpha$$

where α runs over the set of all type (1), (10), (3/2). Note that if $r \in L_-$ with $r^2 = -2$ and $r/2 \in L_-^*$ then $(r + \rho(r))/2 \bmod L_- = \kappa$ (Lemma 7.5). This implies that $H_r \subset \mathcal{B}_\kappa$. Conversely for any (-1)-vector ξ in L_-^* with $\xi \bmod L_- = \kappa$, $\langle \xi, \rho(\xi) \rangle = 0$ and $\xi + \rho(\xi)$ is a (-2)-vector in L_-. Obviously $(\xi + \rho(\xi))/2$ is of type (3/2). Thus we have

$$\mathcal{B}_\kappa = \mathcal{B} \cap \mathcal{D}_{10} = \bigcup_\alpha \mathcal{B}_\alpha$$

where α runs over the set of vectors of type (3/2). We now conclude:

7.7. **Lemma.** $\mathcal{H}/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z})$ *consists of* 63 *components* $\mathcal{H}_\alpha$ (α *is of type* (1)) *and* 36 *components* $\mathcal{H}_\beta$ (β *is of type* (3/2)).

7.8. **Lemma.** *For any α, $\mathcal{H}_\alpha$ is a branch divisor of the covering $\mathcal{B} \to \mathcal{B}/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z})$. If α is of type (1), then the branch degree is two. If α is of type $(3/2)$, the branch degree is four.*

Proof. We use Lemma 7.6. If α is of type (1), then there exists $r \in L_-$ with $r^2 = -2$ such that $\alpha = (r + \rho(r))/2 \bmod L_-$. Then $R^2_{r,\sqrt{-1}}$ is contained in $\tilde{\Gamma}$ and fixes H_r. On the other hand, α is type $(3/2)$, there exists $r \in L_-$ with $r^2 = -2$ such that $\alpha = r/2 \bmod L_-$. Then $R_{r,\sqrt{-1}}$ is contained in $\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z}$ and fixes H_r. Thus the assertion follows. $\qquad\square$

7.9. **Generalized del Pezzo surfaces.** Let $\pi : \hat{P}^7_2 \to P^7_2$ be the blow up of P^7_2 along the 21 subvarieties of codimension 2 corresponding to 21 roots $e_i - e_j$ (see Lemma 4.4). Let $(\hat{P}^7_2)_0$ be the open set of $\hat{P}^7_2$ consisting of the inverse image of ordered 7 points in general position. Let $(\hat{P}^7_2)_1$ be the complement of the inverse images of 36 subvarieties corresponding to the conditions given in Lemma 4.3. Recall that $\mathrm{Sp}(6, \mathbb{F}_2)$ naturally acts on P^7_2 as birational automorphisms ([DO], Chap. VII, §4). It follows from Proposition 5.8 that $(\hat{P}^7_2)_0$ is isomorphic to $(\mathcal{B} \setminus \mathcal{H})/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z})$. We denote this isomorphism by p. Note that p is $\mathrm{Sp}(6, \mathbb{F}_2)$-equivariant.

Let $p_1, ..., p_7$ be an ordered 7 points in $\mathbb{P}^2$ representing a point in $(\hat{P}^7_2)_1 \setminus (\hat{P}^7_2)_0$. For simplicity, we consider generic case and we may assume that $p_1, ..., p_6$ lies on a conic Q. Then after blowing up at 7 points, we get a *generalized* del Pezzo surface S containing a (-2)-curves, i.e., the proper transform of Q. The anti-canonical map of S gives a double cover of $\mathbb{P}^2$ branched along a nodal quartic curve C. In [Kon1], §4, we studied the $K3$ surface X which is the minimal model of the 4-cyclic cover of $\mathbb{P}^2$ branched along C. The Picard lattice of a generic X is isomorphic to $U \oplus A_1^{\oplus 8}$. Moreover X has an elliptic fibration with 8 singular fibers of type III in the sense of Kodaira. This elliptic fibration is induced from the pencil of lines in $\mathbb{P}^2$ through the node of C. 8 singular fibers correspond to two tangent lines of C at the node and six tangent lines of C at some smooth points (note that the normalization of C is a hyperelliptic curve of genus 2). On the other hand, this elliptic fibration is also induced from the pencil of lines on $\mathbb{P}^2$ through p_7. 8 singular fibers correspond to six lines through $p_1, ..., p_6$ and two tangent lines l_1, l_2 of the conic Q. To get X from S, we need a blow up at two points q_1, q_2 at which l_1, l_2 tangent to Q. Note that the $K3$ surface X has new algebraic cycles. Two exceptional curves over q_1, q_2 gives a sublattice $A_1 \oplus A_1$ in L_-. Denote by r_1, r_2 a generator of $A_1 \oplus A_1$. Then $r_2 = -\rho(r_1)$ and (-1)-vector $(r_1 + r_2)/2$ in L_-^* corresponds to $(2\tilde{e}_0 - \tilde{e}_1 - \cdots - \tilde{e}_6)/2$ in Lemma 5.2 under the isomorphism between $L_+^*/L_+ \cong L_-^*/L_-$. The sum of these two (-1)-vectors is represented by the class of (-2)-curve corresponding to the conic Q. Since ω_X is perpendicular to $A_1 \oplus A_1$, the period of the $K3$ surface X is contained in $\mathcal{B}_\alpha$ where $\alpha = (r_1 + r_2)/2 \bmod L_-$. Now we conclude:

7.10. **Proposition.** *p can be extended a holomorphic map*

$$\hat{p} : (\hat{P}^7_2)_1 \to (\mathcal{B} \setminus \mathcal{H}_h)/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z})$$

which is $\mathrm{Sp}(6, \mathbb{F}_2)$-equivariant and isomorphic on the complement of subvarieties of codimension 2.

Proof. Let P be a component from 63 divisors in $\hat{P}^7_2$. Take a general point of P and consider a corresponding 7 points on $\mathbb{P}^2$. As mentioned above, we have a $K3$ surface associated to these 7 points whose period is contained in $\mathcal{B}_\alpha$. Thus, by using the theory of simultaneous resolutions of singularities, p can be extended to a holomorphic map $\hat{p}$ to a general point of P. It follows from the

proof of Theorem 4.3 in [Kon1] that $\hat{p}$ is injective on a general point of P. Since p is equivariant under the action of $\mathrm{Sp}(6, \mathbb{F}_2)$, its extension $\hat{p}$ is $\mathrm{Sp}(6, \mathbb{F}_2)$-equivariant, too. $\qquad\qquad\square$

8. WEIL REPRESENTTION

In this section we study an action of $\mathrm{SL}(2, \mathbb{Z})$ on the group ring $\mathbb{C}[A_{L_-}]$ called Weil representation. The following Lemmas 8.1, 8.5 will be used in the next section to show the existence of a 15-dimensional linear system of automorphic forms on $\mathcal{B}$. For simplicity, we sometimes denote the discriminant quadratic form q_{L_-} by q and the discriminant bilinear form b_{L_-} by b. In the following we denote by S, T a generator of $\mathrm{SL}(2, \mathbb{Z})$:

$$(8.1) \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let ρ be the Weil representation of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{C}[A_{L_-}]$ defined by:

$$(8.2) \qquad \rho(T)(e_\alpha) = e^{\pi\sqrt{-1}\, q(\alpha)} e_\alpha, \quad \rho(S)(e_\alpha) = \frac{\sqrt{-1}}{16} \sum_\delta e^{-2\pi\sqrt{-1}\, b(\delta,\alpha)} e_\delta.$$

The action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{C}[A_{L_-}]$ factorizes to the one of $\mathrm{SL}(2, \mathbb{Z}/4\mathbb{Z})$. The conjugacy classes of $\mathrm{SL}(2, \mathbb{Z}/4\mathbb{Z})$ consist of $\pm E, \pm S, \pm T, \pm T^2, ST, (ST)^2$. Let χ_i $(1 \leq i \leq 10)$ be the characters of irreducible representations of $\mathrm{SL}(2, \mathbb{Z}/4\mathbb{Z})$. For the convenience of the reader we give the character table of $\mathrm{SL}(2, \mathbb{Z}/4\mathbb{Z})$ in the Table 1.

	E	$-E$	S	$-S$	T	$-T$	T^2	$-T^2$	ST	$(ST)^2$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	-1	-1	1	1	1	1
χ_3	1	-1	$\sqrt{-1}$	$-\sqrt{-1}$	$\sqrt{-1}$	$-\sqrt{-1}$	-1	1	-1	1
χ_4	1	-1	$-\sqrt{-1}$	$\sqrt{-1}$	$-\sqrt{-1}$	$\sqrt{-1}$	-1	1	-1	1
χ_5	2	2	0	0	0	0	2	2	-1	-1
χ_6	2	-2	0	0	0	0	-2	2	1	-1
χ_7	3	3	1	1	-1	-1	-1	-1	0	0
χ_8	3	3	-1	-1	1	1	-1	-1	0	0
χ_9	3	-3	$-\sqrt{-1}$	$\sqrt{-1}$	$\sqrt{-1}$	$-\sqrt{-1}$	1	-1	0	0
χ_{10}	3	-3	$\sqrt{-1}$	$-\sqrt{-1}$	$-\sqrt{-1}$	$\sqrt{-1}$	1	-1	0	0

TABLE 1.

On the other hand, for each $u \in A_{L_-}$, we denote by m_0 (resp. m_1) the number of vectors $v \in A_{L_-}$ with $b_{L_-}(u, v) \equiv 0$ (resp. $1/2$). Then m_0, m_1 are given in the Table 2.

u	00	00	00	00	00	00	0	0	0	0	0	0	1	1	1	1	1	1
v	00	0	1	10	1/2	3/2	00	0	1	10	1/2	3/2	00	0	1	10	1/2	3/2
m_0	1	63	63	1	56	72	1	31	31	1	24	40	1	31	31	1	32	32
m_1	0	0	0	0	0	0	0	32	32	0	32	32	0	32	32	0	24	40
u	10	10	10	10	10	10	1/2	1/2	1/2	1/2	1/2	1/2	3/2	3/2	3/2	3/2	3/2	3/2
v	00	0	1	10	1/2	3/2	00	0	1	10	1/2	3/2	00	0	1	10	1/2	3/2
m_0	1	63	63	1	0	0	1	27	36	0	28	36	1	35	28	0	28	36
m_1	0	0	0	0	56	72	0	36	27	1	28	36	0	28	35	1	28	36

TABLE 2.

8.1. Lemma. *Let χ be the character of the representation of $SL(2, \mathbb{Z}/4\mathbb{Z})$ on $\mathbb{C}[A_{L_-}]$. Let $\chi = \sum_i m_i \chi_i$ be the decomposition into irreducible characters. Then*

$$\chi = 7\chi_3 + 15\chi_4 + 21\chi_6 + 28\chi_9 + 36\chi_{10}.$$

Proof. By using the Table 2, we can see that the traces of conjugacy classes of $SL(2, \mathbb{Z}/4\mathbb{Z})$ are as follows:

$$tr(E) = 2^8, \ tr(-E) = -2^8, \ tr(S) = 0, \ tr(-S) = 0, \ tr(T) = -16\sqrt{-1},$$
$$tr(-T) = 16\sqrt{-1}, \ tr(T^2) = 0, \ tr(-T^2) = 0, \ tr(ST) = -1, \ tr((ST)^2) = 1.$$

The assertion now follows from the Table 1. $\square$

8.2. Lemma. (i) *The number of totally isotropic subspaces of dimension 2 in A_{L_-} is 315.*

(ii) *The number of totally isotropic subspaces of dimension 3 in A_{L_-} is 135.*

(iii) *Let e be a non-zero isotropic vector. Then there exist 15 totally isotropic subspaces of dimension 2 and 35 totally isotropic subspaces of dimension 3 containing e. There exist 3 totally isotropic subspaces of dimension 3 containing a fixed totally isotropic subspaces of dimension 2.*

Proof. The number of totally isotropic subspaces of dimension 2 or dimension 3 is given by

$$\frac{(2^6 - 1)2(2^4 - 1)}{(2^2 - 1)(2^2 - 2)} = 315$$

or

$$\frac{(2^6 - 1)2(2^4 - 1)2^2(2^2 - 1)}{(2^3 - 1)(2^3 - 2)(2^3 - 2^2)} = 135$$

respectively. The assertion (iii) now follows from the facts that the number of non-zero isotropic vectors is 63 and each totally isotropic subspace of dimension 3 contains exactly 7 totally isotropic subspaces of dimension 2. $\square$

8.3. Definition-Remark. It follows from Lemma 8.1 that there exists a 15-dimensional subspace of $\mathbb{C}[A_{L_-}]$ on which $SL(2, \mathbb{Z}/4\mathbb{Z})$ acts with character χ_4. We denote by W this 15-dimensional representation.

Note that $O(q_{L_-})$ naturally acts on $\mathbb{C}[A_{L_-}]$. Since the action of $SL(2, \mathbb{Z}/4\mathbb{Z})$ on $\mathbb{C}[A_{L_-}]$ and that of $O(q_{L_-})$ commute, $O(q_{L_-})$ acts on W. One can prove that this 15-dimensional representation of $O(q_{L_-})$ is irreducible (e.g. see Dolgachev, Ortland [DO], Chapter IX, the proof of Proposition 9).

8.4. Definition. Let I be a maximal totally isotropic subspace of A_{L_-} and let $V = \langle I, \kappa \rangle$ be a subspace of A_{L_-} generated by I and κ. Let $\alpha \in A_{L_-}$ with $q_{L_-}(\alpha) = 3/2$, $b_{L_-}(\alpha, c) = 0$ for any $c \in I$ (α is unique modulo V). Define

$$M_+ = \{\alpha + c : c \in I\}, \quad M_- = \{\alpha + c + \kappa : c \in I\},$$

and

$$\theta_V = \sum_{\beta \in M_+} e_\beta - \sum_{\beta \in M_-} e_\beta \in \mathbb{C}[A_{L_-}].$$

8.5. Lemma. (i) $\rho(S)(\theta_V) = -\sqrt{-1}\theta_V$, $\rho(T)(\theta_V) = -\sqrt{-1}\theta_V$. *In particular θ_V is contained in* W.

(ii) *For $a \in V$ with $q_{L_-}(a) = 1$, $t_a(\theta_V) = -\theta_V$ where t_a is the transvection associated to a.*

Proof. (i) If $\beta \in M_\pm$, then $q_{L_-}(\beta) = -1/2$, and hence $\rho(T)(\theta_V) = -\sqrt{-1}\theta_V$. Next by definition,

$$\rho(S)(\theta_V) = \frac{\sqrt{-1}}{16} \sum_\delta \left(\sum_{\beta \in M_+} e^{-2\pi\sqrt{-1}\, b(\delta,\beta)} - \sum_{\beta \in M_-} e^{-2\pi\sqrt{-1}\, b(\delta,\beta)} \right) e_\delta.$$

We denote by c_δ the coefficient of e_δ. If $\delta \in M_+$, then $b_{L_-}(\delta, \beta) = -1/2$ for $\beta \in M_+$ and $b_{L_-}(\delta, \beta) \in \mathbb{Z}$ for $\beta \in M_-$. Hence $c_\delta = -2^3 - 2^3 = -2^4$. Similary if $\delta \in M_-$, then $c_\delta = 2^4$.

Now assume $\delta \notin M_\pm$. If $\delta \in V$, we can easily see that $c_\delta = 0$. Hence we assume $\delta \notin V$. First consider the case $b_{L_-}(\delta, \kappa) \in \mathbb{Z}$. Since $V^\perp = V$, there exists $\gamma \in V$ such that $b_{L_-}(\gamma, \delta) \notin \mathbb{Z}$. In this case $I = \delta^\perp \cap I \cup \{\gamma + a : a \in \delta^\perp \cap I\}$. This implies that

$$\sum_{\beta \in M_+} e^{-2\pi\sqrt{-1}\, b(\delta,\beta)} = \sum_{\beta \in M_-} e^{-2\pi\sqrt{-1}\, b(\delta,\beta)} = 0.$$

Finally if $b_{L_-}(\delta, \kappa) \notin \mathbb{Z}$, then $\delta = \alpha + \delta'$, $b_{L_-}(\delta', \kappa) \in \mathbb{Z}$. Then this case reduces to the previous case.

(ii) The transvection t_a interchanges M_+ and M_- and hence the assertion follows. $\quad\square$

8.6. Lemma. *Let A be a totally isotropic subspace of dimension 2 and let I_i $(i = 1, 2, 3)$ be totally isotropic subspaces of dimension 3 contining A. Let $V_i = \langle I_i, \kappa \rangle$. Then*

$$\theta_{V_1} \pm \theta_{V_2} \pm \theta_{V_3} = 0$$

where the sign $\pm$ are taken for suitable one.

Proof. We fix a decomposition

$$A_{L_-} = u_1 \oplus u_2 \oplus u_3 \oplus q_1 \oplus q_2$$

where u_i is a hyperbolic plane and both q_1 and q_2 are the discriminant quadratic form of A_1. Let e_i, f_i be a basis of u_i $(i = 1, 2)$ with $q_{L_-}(e_i) = q_{L_-}(f_i) = 0$, $b_{L_-}(e_i, f_i) = 1/2$ and α_i is a generator of q_i with $q_{L_-}(\alpha_i) = 3/2$. Then $\kappa = \alpha_1 + \alpha_2$ and

$$A'_{L_-} = u_1 \oplus u_2 \oplus u_3 \oplus \langle \kappa \rangle.$$

We may assume that A is generated by e_1, e_2. Then

$$I_1 = \langle e_1, e_2, e_3 \rangle, I_2 = \langle e_1, e_2, f_3 \rangle, I_3 = \langle e_1, e_2, e_3 + f_3 + \kappa \rangle.$$

We take α_1 as α for I_1, I_2 and $\alpha_1 + e_3$ for I_3. An easy calculation shows that $\theta_{V_1} - \theta_{V_2} = \theta_{V_3}$. $\quad\square$

8.7. Lemma. *Let e be a non-zero isotropic vector. Among 15 totally isotropic subspaces of dimension 3 containing e, there are six I_j $(j = 1, ..., 6)$ such that for other I, θ_V $(V = \langle I, \kappa \rangle)$ is a linear combination of θ_{V_j} $(V_j = \langle I_j, \kappa \rangle)$, and the set of isotropic vectors in $I_1 \cup I_2 \cup I_3$ coincides with that of $I_4 \cup I_5 \cup I_6$ including multiplicities.*

Proof. We use the same notation as in Lemma 8.6 and assume that $e = e_1$. Then by using Lemma 8.6, we can see that the following $I_1, ..., I_6$ satisfies that θ_V is a linear combination of $\theta_{V_1}, ..., \theta_{V_6}$:

$$I_1 = \langle e_1, e_2, e_3 \rangle, \ I_2 = \langle e_1, f_2, f_3 \rangle, \ I_3 = \langle e_1, e_2 + f_3, f_2 + e_3 \rangle,$$

$$I_4 = \langle e_1, e_2, f_3 \rangle, \ I_5 = \langle e_1, f_2, e_3 \rangle, \ I_6 = \langle e_1, e_2 + e_3, f_2 + f_3 \rangle.$$

We can easily see that both $I_1 \cup I_2 \cup I_3$ and $I_4 \cup I_5 \cup I_6$ contain 0 and e_1 with multiplicity 3 and other isotorpic vectors with multiplicity 1.

$\square$

9. AUTOMORPHIC FORMS: ADDITIVE LIFTINGS

In this section we shall construct a linear system of meromorphic automorphic forms by applying Borcherds theory [B1].

Let $\rho : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[A_{L_-}])$ be the Weil representation (8.2). A holomorphic map

$$f : H^+ \to \mathbb{C}[A_{L_-}]$$

is called a *vector valued modular form of weight k with respect to ρ* if

$$f(M\tau) = \rho(M)(c\tau + d)^k f(\tau)$$

for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. For any $\theta \in W$, define

$$f_\theta(\tau) = \eta(\tau)^{-6}\theta$$

where η is the Dedekind eta function. Then it follows that

$$f_\theta(\tau + 1) = \eta(\tau + 1)^{-6}\theta = -\sqrt{-1}\eta(\tau)^{-6}\theta = \rho(T)f_\theta(\tau),$$

$$f_\theta(-1/\tau) = \eta(-1/\tau)^{-6}\theta = -\sqrt{-1}\tau^{-3}\eta(\tau)^{-6}\theta = \tau^{-3}\rho(S)f_\theta(\tau),$$

that is, $f_\theta(\tau)$ is a vector valued modular form of weight -3 with respect to ρ. Thus we have a 15-dimensional space $\tilde{W}$ of modular forms of weight -3 and of type ρ.

By applying Borcherds [B1], Theorem 14.3, we have a $\mathrm{O}(q_{L_-})$-equivariant map

$$F : W \to \tilde{W} \to W_2(\mathcal{D}, \tilde{\mathrm{O}}(L_-))$$

where $W_2(\mathcal{D}, \tilde{\mathrm{O}}(L_-))$ is the space of meromorphic automorphic forms of weight 2 on a 12-dimensional bounded symmetric domain $\mathcal{D}$ associated to L_- with respect to $\tilde{\mathrm{O}}(L_-) = \mathrm{Ker}\{\mathrm{O}(L_-) \to \mathrm{O}(q_{L_-})\}$. For θ_V in Definition 8.4, we denote f_{θ_V} by f_V for simplicity, and define $F_V = F(\theta_V)$. Let $\alpha \in V$ be a non isotropic vector. Take $r \in L_-$ with $r^2 = -2$ and $\alpha = (r + \rho(r))/2 \mod L_-$. Then the reflection $s_{(r+\rho(r))}$ fixes the hyperplane $(r + \rho(r))^\perp$ and it induces a transvection t_α on A_{L_-} (see 6.2). It now follows from Lemma 8.5, (ii) and the $\mathrm{O}(q_{L_-})$-equivariantness of F that F_V vanishes along 8 Heegner divisors associated to non isotoropic vectors in V. Note that the negative power of the Fourier expansion of f_V is $q^{-1/4}\theta_V$ $(q = e^{2\pi\sqrt{-1}\tau})$. It follows from Borcherds [B1], Theorem 14.3 (also see

its proof in page 555) that F_V has poles with multiplicity 2 along Heegner divisors associated to 16 vectors with norm $-1/2$ appeared in θ_V.

9.1. **Lemma.** *The additive lifting F is injective.*

Proof. The map F is $\mathrm{O}(q_{L_-})$-equivariant and W is an irreducible representation of $\mathrm{O}(q_{L_-})$. Since F_V has a pole, $F_V \neq 0$. The assertion now follows from the Schur's Lemma. $\qquad\square$

We now conclude:

9.2. **Theorem.** *For each V, F_V is a meromorphic automorphic form on $\mathcal{D}$ of weight 2 vanishing at least along Heegner divisors associated to 8 non isotropic vectors in V and with poles of multiplicity 2 along Heegner divisors associated to 16 vectors with norm $3/2$. The product of 135 F_V's is a meromorphic automorphic form on $\mathcal{D}$ of weight 270 vanishing along the Heegner divisor $\mathcal{D}_1$ with at least multiplicity 15 and along the Heegner divisor $\mathcal{D}_{10}$ with at least multiplicity 135 and having poles along the Heegner divisor $\mathcal{D}_{3/2}$ with multiplicity 60 exactly.*

Proof. Each V contains 7 non-isotropic vectors of type (1) and the number of vectors of type (1) is 63. Hence the product vanishes along each Heegner divisor of type (1) with at least multiplicity

$$\frac{135 \times 7}{63} = 15.$$

Each V contains κ and hence the product vanishes along the Heegner divisor $\mathcal{D}_{10}$ with at least multiplicity 135. Finally the pole order of the product along $\mathcal{D}_{3/2}$ is given by

$$\frac{135 \times 16 \times 2}{72} = 60.$$

$\qquad\square$

10. Automorphic forms: Borcherds products

In the previous section we construct a linear system of automorphic forms. To determind the divisor of F_V we need an another automorphic form with known zeros and poles. In this section we construct a suitable such automorphic form by applying Borcherds theory [B1], [B2], [F].

10.1. **Obstruction.** Borcherds products are meromorphic automorphic forms on $\mathcal{D}$ whose zeros and poles lie on Heegner divisors. To show the existence of some Borcherds products, we introduce the *obstraction space* consisting of all vector valued elliptic modular forms $\{f_\alpha\}_{\alpha \in A_{L_-}}$ of weight $(2+12)/2 = 7$ and with respect to the dual representation of ρ:

$$(10.1) \qquad f_\alpha(\tau + 1) = e^{-\pi\sqrt{-1}\,q(\alpha)} f_\alpha(\tau), \quad f_\alpha(-1/\tau) = -\frac{\sqrt{-1}\tau^7}{2^4} \sum_\beta e^{2\pi\sqrt{-1}\,b(\alpha,\beta)} f_\beta(\tau).$$

We shall apply the next theorem to show the existence of some Borcherds products (see Theorem 10.8, Corollary 10.9).

10.2. **Theorem.** (Borcherds [B2], Freitag [F], Theorem 5.2) *A linear combination*

$$\sum_{\alpha \in A_{L_-}, n<0} c_{\alpha,n} \mathcal{D}_{\alpha,n}, \ c_{\alpha,n} \in \mathbb{Z}$$

is the divisor of a meromorphic automorphic form on $\mathcal{D}$ of weight k if for every cusp form

$$f = \{f_\alpha(\tau)\}_{\alpha \in A_{L_-}}, \ f_\alpha(\tau) = \sum_{n \in \mathbb{Q}} a_{\alpha,n} e^{2\pi\sqrt{-1}n\tau}$$

in the obstruction space, the relation

$$\sum_{\alpha \in A_{L_-}, n<0} a_{\alpha,-n/2} c_{\alpha,n} = 0$$

holds. In this case the weight k is given by

$$k = \sum_{\alpha \in A_{L_-}, n \in \mathbb{Z}} b_{\alpha,n/2} c_{\alpha,-n}$$

where $b_{\alpha,n}$ are the Fourier coefficients of the Eisenstein series in the obstruction space with the constant term $b_{0,0} = -1/2$ and $b_{\alpha,0} = 0$ for $\alpha \neq 0$.

In the following we shall study the divisors $\sum_{\alpha \in A_{L_-}, n<0} c_{\alpha,n} \mathcal{D}_{\alpha,n}$ where $c_{\alpha,n}$ depends only on the type of α. We denote by

$$h_{00}, \ h_0, \ h_1, \ h_{10}, \ h_{1/2}, \ h_{3/2}$$

the sum of f_α's according to their types. Then we need to consider the 6-dimensional representation V given by:

(10.2)
$$\rho^*(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{-1} \end{pmatrix};$$

(10.3)
$$\rho^*(S) = -\frac{\sqrt{-1}}{16} \begin{pmatrix} 1 & 63 & 63 & 1 & 56 & 72 \\ 1 & -1 & -1 & 1 & -8 & 8 \\ 1 & -1 & -1 & 1 & 8 & -8 \\ 1 & 63 & 63 & 1 & -56 & -72 \\ 1 & -9 & 9 & -1 & 0 & 0 \\ 1 & 7 & -7 & -1 & 0 & 0 \end{pmatrix}.$$

10.3. **Lemma.** *The dimension of the space of modular forms of weight 7 and of type ρ^* is 4. The dimension of the space of Eisenstein forms of weight 7 and of type ρ^* is 2.*

Proof. In general, the dimension of the space of modular forms of weight $k > 2$ and of type ρ^* is given by

$$d + dk/12 - \alpha(e^{k\pi\sqrt{-1}/2}\rho^*(S)) - \alpha((e^{k\pi\sqrt{-1}/3}\rho^*(ST))^{-1}) - \alpha(\rho^*(T))$$

([B2], section 4, [F], Proposition 2.1). Here $d = \dim\{x \in V : \rho^*(-E)x = (-1)^k x\}$ and

$$\alpha(A) = \sum_\lambda \alpha$$

where λ runs through all eigenvalues of A and $\lambda = e^{2\pi\sqrt{-1}\alpha}, 0 \le \alpha < 1$.

In our situation, $k = 7$ and $d = \dim(V) = 6$. An elementary calculation shows that

$$\alpha(e^{7\pi\sqrt{-1}/2}\rho^*(S)) = 3/2, \ \alpha((e^{k\pi\sqrt{-1}/3}\rho^*(ST))^{-1}) = 2, \ \alpha(\rho^*(T)) = 2.$$

On the other hand, the space of Eisenstein series is isomorphic to the subspace of V given by

$$\rho^*(T)(x) = x, \ \rho^*(-E)(x) = (-1)^k x$$

(see Remark 2.2 in [F]). Thus we have the assertion. $\qquad\qquad\square$

10.4. **Eisenstein forms.** In the following we shall calculate Eisenstein forms $\{h_\alpha\}$ of weight 7 and of type ρ^*. Then $\{h_\alpha\}$ should satisfy the following:

$$(10.4) \quad \begin{cases} h_{00}(\tau+1) = h_{00}(\tau), \ h_0(\tau+1) = h_0(\tau), \ h_1(\tau+1) = -h_1(\tau), \\ h_{10}(\tau+1) = -h_{10}(\tau), \ h_{1/2}(\tau+1) = -\sqrt{-1}h_{1/2}(\tau), \ h_{3/2}(\tau+1) = \sqrt{-1}h_{3/2}(\tau), \\ h_{00}(-1/\tau) = \frac{-\sqrt{-1}\tau^7}{16}(h_{00} + h_0 + h_1 + h_{10} + h_{1/2} + h_{3/2}), \\ h_0(-1/\tau) = \frac{-\sqrt{-1}\tau^7}{16}(63h_{00} - h_0 - h_1 + 63h_{10} - 9h_{1/2} + 7h_{3/2}), \\ h_1(-1/\tau) = \frac{-\sqrt{-1}\tau^7}{16}(63h_{00} - h_0 - h_1 + 63h_{10} + 9h_{1/2} - 7h_{3/2}), \\ h_{10}(-1/\tau) = \frac{-\sqrt{-1}\tau^7}{16}(h_{00} + h_0 + h_1 + h_{10} - h_{1/2} - h_{3/2}), \\ h_{1/2}(-1/\tau) = \frac{-\sqrt{-1}\tau^7}{16}(56h_{00} - 8h_0 + 8h_1 - 56h_{10}), \\ h_{3/2}(-1/\tau) = \frac{-\sqrt{-1}\tau^7}{16}(72h_{00} + 8h_0 - 8h_1 - 72h_{10}). \end{cases}$$

Let

$$E_1 = G_7^{(0,1)}(\tau), \ E_2 = G_7^{(1,0)}(\tau), \ E_3 = G_7^{(1,1)}(\tau),$$
$$E_4 = G_7^{(1,2)}(\tau), \ E_5 = G_7^{(1,3)}(\tau), \ E_6 = G_7^{(2,1)}(\tau)$$

be the Eisenstein series of weight 7 and level 4. Here $(0,1),(1,0),(1,1),(1,2),(1,3),(2,1) \in (\mathbb{Z}/2\mathbb{Z})^2$. Then their Fourier expansion are given as follows (Koblitz, [Kob], Chap. III, §3, Proposition 22):

$$(10.5) \quad \begin{cases} E_1 = \frac{(2\pi)^7}{2\cdot 7!}B_7(1/4) + \frac{(-2\pi\sqrt{-1})^7 2\sqrt{-1}}{4^7\cdot 6!}q + \cdots; \\ E_2 = \frac{(-2\pi\sqrt{-1})^7}{4^7\cdot 6!}(q^{1/4} + 2^6 q^{1/2} + (3^6 - 1)q^{3/4} + 4^6 q + \cdots); \\ E_3 = \frac{(-2\pi\sqrt{-1})^7}{4^7\cdot 6!}(\sqrt{-1}q^{1/4} - 2^6 q^{1/2} - \sqrt{-1}(3^6 - 1)q^{3/4} + 4^6 q + \cdots); \\ E_4 = \frac{(-2\pi\sqrt{-1})^7}{4^7\cdot 6!}(-q^{1/4} + 2^6 q^{1/2} - (3^6 - 1)q^{3/4} + 4^6 q + \cdots); \\ E_5 = \frac{(-2\pi\sqrt{-1})^7}{4^7\cdot 6!}(-\sqrt{-1}q^{1/4} - 2^6 q^{1/2} + \sqrt{-1}(3^6 - 1)q^{3/4} + 4^6 q + \cdots); \\ E_6 = \frac{(-2\pi\sqrt{-1})^7}{4^7\cdot 6!}(2\sqrt{-1}q^{1/2} + 0 \cdot q + \cdots) \end{cases}$$

where B_7 is the Bernoulli polynomial. The action of S, T on E_i is as follows:

$$T(E_1) = E_1, \ T : E_2 \to E_3 \to E_4 \to E_5 \to E_2, \ T(E_6) = -E_6;$$
$$S : E_1 \to E_2 \to -E_1, \ E_3 \to E_5 \to -E_3, \ E_4 \to -E_6 \to -E_4.$$

Then the Eisenstein form is given by:

$$(10.6) \quad \begin{cases} h_{00} = \alpha E_1 + \frac{\sqrt{-1}}{16}(\alpha + 63\beta)(E_2 + E_3 + E_4 + E_5), \\ h_0 = \beta E_1 + \frac{63\sqrt{-1}}{16}(\alpha - \beta)(E_2 + E_3 + E_4 + E_5), \\ h_1 = \frac{63\sqrt{-1}}{16}(\alpha - \beta)(E_2 - E_3 + E_4 - E_5) + \beta E_6, \\ h_{10} = \frac{\sqrt{-1}}{16}(\alpha - \beta)(E_2 - E_3 + E_4 - E_5) + \alpha E_6, \\ h_{1/2} = \frac{56\sqrt{-1}}{16}(\alpha - 9\beta)(E_2 + \sqrt{-1}E_3 - E_4 - \sqrt{-1}E_5), \\ h_{3/2} = \frac{72\sqrt{-1}}{16}(\alpha + 7\beta)(E_2 - \sqrt{-1}E_3 - E_4 + \sqrt{-1}E_5) \end{cases}$$

where α, β are parameters. We need the solution such that the constant terms of $h_{00}, h_0, h_{10}, h_1, h_{1/2}, h_{3/2}$ are $-1/2, 0, 0, 0, 0, 0$, respectively. Such Eisenstein form is given by:

$$(10.7) \quad \begin{cases} h_{00} = \alpha E_1 + \frac{\sqrt{-1}\alpha}{16}(E_2 + E_3 + E_4 + E_5) = -\frac{1}{2} + \frac{2^{10}+2}{61}q + \cdots, \\ h_0 = \frac{63\sqrt{-1}\alpha}{16}(E_2 + E_3 + E_4 + E_5) = \frac{4^5 \cdot 63}{61}q + \cdots, \\ h_1 = \frac{63\sqrt{-1}\alpha}{16}(E_2 - E_3 + E_4 - E_5) = \frac{2^4 \cdot 63}{61}q^{1/2} + \cdots, \\ h_{10} = \frac{\sqrt{-1}\alpha}{16}(E_2 - E_3 + E_4 - E_5) + \alpha E_6 = \frac{18}{61}q^{1/2} + \cdots, \\ h_{1/2} = \frac{56\sqrt{-1}\alpha}{16}(E_2 + \sqrt{-1}E_3 - E_4 - \sqrt{-1}E_5) = \frac{2^4 \cdot 7^2 \cdot 13}{61}q^{3/4} + \cdots, \\ h_{3/2} = \frac{72\sqrt{-1}\alpha}{16}(E_2 - \sqrt{-1}E_3 - E_4 + \sqrt{-1}E_5) = \frac{18}{61}q^{1/4} + \cdots \end{cases}$$

where

$$\alpha = -\frac{7!}{(2\pi)^7 B_7(1/4)} = \frac{-4^7 \cdot 6!}{(2\pi)^7 61}.$$

10.5. Cusp forms.
The obstruction space has dimension 4 and it contains 2-dimensional subspace of cusp forms (Lemma 10.3). We shall calculate cusp forms in the obstruction space. We consider the following two types:

(A) $\{\eta^6(\tau)g_\gamma\}$;

(B) $\{\eta^{12}(\tau)g_\gamma\}$.

10.6. Case (A).
We denote by

$$F_1 = G_4^{(0,1)}(\tau), \ F_2 = G_4^{(1,0)}(\tau), \ F_3 = G_4^{(1,1)}(\tau),$$

$$F_4 = G_4^{(1,2)}(\tau), \ F_5 = G_4^{(1,3)}(\tau), \ F_6 = G_4^{(2,1)}(\tau)$$

the Eisenstein series of weight 4 and level 4. Their Fourier expansion are as follows (Koblitz, [Kob], Chap. III, §3, Proposition 22):

$$(10.8) \quad \begin{cases} F_1 = c_4(1 - 2^4 q^2 + \cdots); \\ F_2 = c_4(q^{1/4} + 2^3 q^2 + (1 + 3^3)q^{3/4} + 4^3 q + \cdots); \\ F_3 = c_4(\sqrt{-1}q^{1/4} - 2^3 q^2 - \sqrt{-1}(1 + 3^3)q^{3/4} + 4^3 q + \cdots); \\ F_4 = c_4(-q^{1/4} + 2^3 q^2 - (1 + 3^3)q^{3/4} + 4^3 q + \cdots); \\ F_5 = c_4(-\sqrt{-1}q^{1/4} - 2^3 q^2 + \sqrt{-1}(1 + 3^3)q^{3/4} + 4^3 q + \cdots); \\ F_6 = c_4(-2^4 q + \cdots), \end{cases}$$

where $c_4 = \frac{(2\pi)^4}{2^6 \cdot 4!}$. The acion of S, T on F_i is as follows:

$$T(F_i) = F_i, i = 1, 6, \ T : F_2 \to F_3 \to F_4 \to F_5 \to F_2;$$

$$S : F_1 \to F_2 \to F_1, \ F_3 \to F_5 \to F_3, \ F_4 \to F_6 \to F_4.$$

Recall that $\eta^6(\tau + 1) = \sqrt{-1}\eta^6(\tau)$ and $\eta^6(-1/\tau) = \sqrt{-1}\tau^3\eta^6(\tau)$. If we write $h_\alpha = \eta^6(\tau)g_\alpha$, we need to find $\{g_\alpha\}$ satisfying:

$$(10.9) \quad
\begin{cases}
g_{00}(\tau + 1) = -\sqrt{-1}g_{00}(\tau), \ g_0(\tau + 1) = \sqrt{-1}g_0(\tau), \ g_1(\tau + 1) = \sqrt{-1}h_1(\tau), \\
g_{10}(\tau + 1) = \sqrt{-1}g_{10}(\tau), \ g_{1/2}(\tau + 1) = -g_{1/2}(\tau), \ g_{3/2}(\tau + 1) = g_{3/2}(\tau), \\
g_{00}(-1/\tau) = \frac{-\tau^4}{16}(g_{00} + g_0 + g_1 + g_{10} + g_{1/2} + g_{3/2}), \\
g_0(-1/\tau) = \frac{-\tau^4}{16}(63g_{00} - g_0 - g_1 + 63g_{10} - 9g_{1/2} + 7g_{3/2}), \\
g_1(-1/\tau) = \frac{-\tau^4}{16}(63g_{00} - g_0 - g_1 + 63g_{10} + 9g_{1/2} - 7g_{3/2}), \\
g_{10}(-1/\tau) = \frac{-\tau^4}{16}(g_{00} + g_0 + g_1 + g_{10} - g_{1/2} - g_{3/2}), \\
g_{1/2}(-1/\tau) = \frac{-\tau^4}{16}(56g_{00} - 8g_0 + 8g_1 - 56g_{10}), \\
g_{3/2}(-1/\tau) = \frac{-\tau^4}{16}(72g_{00} + 8g_0 - 8g_1 - 72g_{10}).
\end{cases}$$

Then one dimensional subspace of cusp forms is given by

$$(10.10) \quad
\begin{cases}
h_{00} = -\frac{\beta\eta(\tau)^6}{16}(F_2 + \sqrt{-1}F_3 - F_4 - \sqrt{-1}F_5) = -\frac{\beta c_4}{16}(4(3^3 + 1)q + \cdots), \\
h_0 = -\frac{7\beta\eta(\tau)^6}{16}(F_2 + \sqrt{-1}F_3 - F_4 - \sqrt{-1}F_5) = -\frac{7\beta c_4}{16}(4(3^3 + 1)q + \cdots), \\
h_1 = \frac{7\beta\eta(\tau)^6}{16}(F_2 - \sqrt{-1}F_3 - F_4 + \sqrt{-1}F_5) = \frac{7\beta c_4}{16}(4q^{1/2} + \cdots), \\
h_{10} = \frac{\beta\eta(\tau)^6}{16}(F_2 - \sqrt{-1}F_3 - F_4 + \sqrt{-1}F_5) = \frac{\beta c_4}{16}(4q^{1/2} + \cdots), \\
h_{1/2} = 0, \\
h_{3/2} = \beta\eta(\tau)^6(F_1 - F_6) = \beta c_4(q^{1/4} + \cdots)
\end{cases}$$

where β is a parameter.

10.7. **Case (B).** Let
$$G_1 = G_1^{(0,1)}(\tau), \ G_2 = G_1^{(1,0)}(\tau), \ G_3 = G_1^{(1,1)}(\tau),$$

$$G_4 = G_1^{(1,2)}(\tau), \ G_5 = G_1^{(1,3)}(\tau), \ G_6 = G_1^{(2,1)}(\tau)$$

be the Eisenstein series of weight 1 and level 4.

The action of S, T on G_i is:

$$T(G_1) = G_1, G_2 \to G_3 \to G_4 \to G_5 \to G_2, T(G_6) = -G_6;$$

$$S : G_1 \to G_2 \to -G_1, G_3 \to G_5 \to -G_3, G_6 \to G_4 \to -G_6.$$

Write $h_\alpha = \eta^{12}(\tau)g_\alpha$.

Since $\eta^{12}(\tau + 1) = -\eta^{12}(\tau), \eta^{12}(-1/\tau) = -\tau^6\eta^{12}(\tau)$, $\{g_\alpha\}$ satisfies the following:

$$(10.11) \begin{cases} g_{00}(\tau+1) = -g_{00}(\tau), \ g_0(\tau+1) = -g_0(\tau), \ g_1(\tau+1) = g_1(\tau), \\ g_{10}(\tau+1) = g_{10}(\tau), \ g_{1/2}(\tau+1) = \sqrt{-1}g_{1/2}(\tau), \ g_{3/2}(\tau+1) = -\sqrt{-1}g_{3/2}(\tau), \\ g_{00}(-1/\tau) = \frac{\sqrt{-1}\tau}{16}(g_{00} + g_0 + g_1 + g_{10} + g_{1/2} + g_{3/2}), \\ g_0(-1/\tau) = \frac{\sqrt{-1}\tau}{16}(63g_{00} - g_0 - g_1 + 63g_{10} - 9g_{1/2} + 7g_{3/2}), \\ g_1(-1/\tau) = \frac{\sqrt{-1}\tau}{16}(63g_{00} - g_0 - g_1 + 63g_{10} + 9g_{1/2} - 7g_{3/2}), \\ g_{10}(-1/\tau) = \frac{\sqrt{-1}\tau}{16}(g_{00} + g_0 + g_1 + g_{10} - g_{1/2} - g_{3/2}), \\ g_{1/2}(-1/\tau) = \frac{\sqrt{-1}\tau}{16}(56g_{00} - 8g_0 + 8g_1 - 56g_{10}), \\ g_{3/2}(-1/\tau) = \frac{\sqrt{-1}\tau}{16}(72g_{00} + 8g_0 - 8g_1 - 72g_{10}). \end{cases}$$

$\{G_i\}$ are not linearly independent, but there are no cusp forms of weight 1 and level 4. Hence g_α can be written as a linear combination of $\{G_i\}$. Then solutions are

$$(10.12) \begin{cases} g_{00} = -\frac{\sqrt{-1}(\alpha+\beta)}{16}(G_2 - G_3 + G_4 - G_5) + \alpha G_6, \\ g_0 = -\frac{\sqrt{-1}(63\alpha-\beta)}{16}(G_2 - G_3 + G_4 - G_5) + \beta G_6, \\ g_1 = \beta G_1 - \frac{\sqrt{-1}(63\alpha-\beta)}{16}(G_2 + G_3 + G_4 + G_5), \\ g_{10} = \alpha G_1 - \frac{\sqrt{-1}(\alpha+\beta)}{16}(G_2 + G_3 + G_4 + G_5), \\ g_{1/2} = \frac{\sqrt{-1}(56\alpha-8\beta)}{16}(G_2 - \sqrt{-1}G_3 - G_4 + \sqrt{-1}G_5), \\ g_{3/2} = \frac{\sqrt{-1}(72\alpha+8\beta)}{16}(G_2 + \sqrt{-1}G_3 - G_4 - \sqrt{-1}G_5). \end{cases}$$

The Fourier coefficients of G_i are as follows (Schoenberg [S], Chap. VII, §2):

$$(10.13) \begin{cases} G_1 = a_0 + \pi q + \cdots, \\ G_2 = b_0 - \frac{\pi\sqrt{-1}}{2}(q^{1/4} + q^{1/2} + q + \cdots), \\ G_3 = b_0 - \frac{\pi\sqrt{-1}}{2}(\sqrt{-1}q^{1/4} - q^{1/2} + q + \cdots), \\ G_4 = b_0 - \frac{\pi\sqrt{-1}}{2}(-q^{1/4} + q^{1/2} + q + \cdots), \\ G_5 = b_0 - \frac{\pi\sqrt{-1}}{2}(-\sqrt{-1}q^{1/4} - q^{1/2} + q + \cdots), \\ G_6 = c_0 + \pi q^{1/2} + \cdots, \end{cases}$$

where

$$a_0 = \frac{1}{4}\lim_{s\to 0}[\zeta(1+s, 1/4) - \zeta(1+s, 3/4)],$$

$$b_0 = \frac{-\pi\sqrt{-1}}{4}[\zeta(0, 1/4) - \zeta(0, 3/4)],$$

$$c_0 = \frac{-\pi\sqrt{-1}}{4}[\zeta(0, 1/2) - \zeta(0, 1/2)] = 0.$$

Since $\zeta(0, a) = 1/2 - a$ (Whittaker, Watson [W], 13.21), $b_0 = -\pi\sqrt{-1}/8$. On the other hand,

$$\lim_{s\to 0}[\zeta(s+1, a) - 1/s] = -\frac{\Gamma'(a)}{\Gamma(a)}$$

(Whittaker, Watson, [W], 13.21). Hence

$$a_0 = \frac{\Gamma(1/4)\Gamma'(3/4) - \Gamma'(1/4)\Gamma(3/4)}{4\Gamma(1/4)\Gamma(3/4)}.$$

By $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$ (Whittaker, Watson [W], 12.14), we have $a_0 = \pi/4$. Thus we have

(10.14)
$$\begin{cases} h_{00} = \eta(\tau)^{12}g_{00} = \frac{\pi(7\alpha-\beta)}{8}q + \cdots, \\ h_0 = \eta(\tau)^{12}g_0 = -\frac{9\pi(7\alpha-\beta)}{8}q + \cdots, \\ h_1 = \eta(\tau)^{12}g_1 = -\frac{9\pi(7\alpha-\beta)}{32}q^{1/2} + \cdots, \\ h_{10} = \eta(\tau)^{12}g_{10} = \frac{\pi(7\alpha-\beta)}{32}q^{1/2} + \cdots, \\ h_{1/2} = \eta(\tau)^{12}g_{1/2} = \pi(7\alpha - \beta)q^{3/4} + \cdots, \\ h_{3/2} = \eta(\tau)^{12}g_{3/2} = 0 \cdot q^{1/4} + \cdots, \end{cases}$$

10.8. **Theorem.** *Let*

$$D = m_1\mathcal{D}_1 + m_{10}\mathcal{D}_{10} + m_{1/2}\mathcal{D}_{1/2} + m_{3/2}\mathcal{D}_{3/2}.$$

Then D is a divisor of meromorphic automorphic form F on $\mathcal{D}$ if

$$7m_1 + m_{10} + 4m_{3/2} = 0, \quad -9m_1 + m_{10} + 32m_{1/2} = 0,$$

that is

$$m_{10} = 9m_1 - 32m_{1/2}, \quad m_{3/2} = 8m_{1/2} - 4m_1.$$

In this case, the weight of F is given by

$$2 \cdot 3^2 m_1 + 2^5 \cdot 5 m_{1/2}.$$

Proof. The first assertion follows from Theorem 10.2 and the equations 10.10, 10.14. By using Theorem 10.2 and the equation 10.7, we can see that the weight of F is given by

$$\frac{2^4 \cdot 63}{61}m_1 + \frac{18}{61}m_{10} + \frac{2^4 \cdot 7^2 \cdot 13}{61}m_{1/2} + \frac{18}{61}m_{3/2} = 2 \cdot 3^2 m_1 + 2^5 \cdot 5 m_{1/2}.$$

□

10.9. **Corollary.** *There exists a meromorphic automorphic forms Φ on $\mathcal{D}$ of weight 18 whose divisor is*

$$\mathcal{D}_1 + 9\mathcal{D}_{10} - 4\mathcal{D}_{3/2}.$$

Proof. Put $m_1 = 1, m_{1/2} = 0$ in Theorem 10.8. □

10.10. **Theorem.** *Let I be a maximal totally isotropic subspace and let $V = \langle I, \kappa \rangle$ as in Definition 8.4. Let F_V be the aditive lifting corresponding to V. Then the divisor of F_V is*

$$\sum_{a \in V, \, q(a)=1} \mathcal{D}_a - \sum_{c \in I}(2\mathcal{D}_{c+\alpha} + 2\mathcal{D}_{c+\alpha+\kappa}).$$

Proof. Consider the product Ψ of all 135 F_V which has weight $135 \cdot 2$. It follows from Theorem 9.2 that Ψ vanishes along $\mathcal{D}_1$ with at least multiplicity 15 and along $\mathcal{D}_{10}$ with at least multiplicity 135 and has poles along $\mathcal{D}_{3/2}$ with multiplicity 60. Then the quotient Ψ/Φ^{15} is a holomorphic automorphic form of weight zero (Corollary 10.9). Now the Koeher principle implies that Ψ/Φ^{15} is constant. □

10.11. Corollary. *The* 135 F_V *satisfies* 63 *cubic relations corresponding to* 63 *non-zero isotropic vectors.*

Proof. We use the notation as in Lemma 8.7. By applying Theorem 10.10, the automorphic forms $F_{V_1} F_{V_2} F_{V_3}$ and $F_{V_4} F_{V_5} F_{V_6}$ have the same divisor and hence

$$F_{V_1} F_{V_2} F_{V_3} = c F_{V_4} F_{V_5} F_{V_6} \ (c \in \mathbb{C}).$$

$\square$

10.12. Corollary. *The restriction of* F_V *to* $\mathcal{B}$ *has the divisor*

$$2\mathcal{B}_\kappa + \sum_{a \in V, \ q(a)=1, a \neq \kappa} 2\mathcal{B}_a - \sum_{c \in I} 4\mathcal{B}_{c+\alpha}.$$

Proof. Note that $\mathcal{D}_{c+\alpha} \cap \mathcal{B} = \mathcal{D}_{c+\alpha+\kappa} \cap \mathcal{B} = \mathcal{B}_{c+\alpha}$ and the fact (7.1). Then the assertion follows. $\square$

By definition, F_V is invariant with respect to $\tilde{\Gamma}$ but not invariant under $\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z}$. However $F_V(\rho(\omega)) = F_V(\sqrt{-1}\omega) = (\sqrt{-1})^2 F_V(\omega) = -F_V(\omega)$ for any V. Hence for totally isotropic subspaces I, I', the quotient $F_V/F_{V'}$ ($V' = \langle I', \kappa \rangle$) is invariant with respect to $\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z}$. Thus we can determine the divisor of $F_V/F_{V'}$ on $\mathcal{B}/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z})$. Let

$$2\mathcal{B}_\kappa + \sum_{a' \in V', \ q(a')=1, a' \neq \kappa} 2\mathcal{B}_{a'} - \sum_{c' \in I'} 4\mathcal{B}_{c'+\alpha'}$$

be the divisor of $F_{V'}$ on $\mathcal{B}$. Let $\pi : \mathcal{B} \to \mathcal{B}/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z})$ be the natural projection. Then π is branched over $\mathcal{H}_\alpha$ and its branch degree is two (resp. four) if α is of type (1) (resp. type $(3/2)$) (Lemma 7.8). Hence the vanishing order or pole order of F_V/F'_V along $\mathcal{H}_\alpha$ is given by dividing the vanishing order or pole order along $\mathcal{B}_\alpha$ by its branch degree. Thus we have:

10.13. Corollary. *The divisor of* $F_V/F_{V'}$ *on* $\mathcal{B}/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z})$ *is given by*

$$\sum_{a \in V, \ q(a)=1, a \neq \kappa} \mathcal{H}_a - \sum_{a' \in V', \ q(a')=1, a' \neq \kappa} \mathcal{H}_{a'} + \sum_{c' \in I'} \mathcal{H}_{c'+\alpha'} - \sum_{c \in I} \mathcal{H}_{c+\alpha}.$$

11. AUTOMORPHIC FORMS AND GÖPEL INVARIANTS

The purpose of this section is to prove the following:

11.1. Theorem. *The linear system defined by* 135 *meromorphic automorphic forms* F_V *gives a rational map*

$$\varphi : \mathcal{B}/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z}) - - \to \mathbb{P}^{14}$$

which is birational onto its image. The image of φ *satisfies* 63 *cubic relations. Under the identification between* P_2^7 *and* $\mathcal{B}/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z})$ *by a birational isomorphism, the map* φ *coincides with* ψ *given by Göpel invariants in Theorem* 4.1.

Proof. We shall show that the map φ coincides with ψ. Then the remaining assertions follow from Theorem 4.2, Corollary 10.11.

Under the isomorphism $\hat{p}$ in Proposition 7.10, we identify $(\hat{P}_2^7)_1$ with $(\mathcal{B} \setminus \mathcal{H}_h)/(\tilde{\Gamma} \cdot \mathbb{Z}/2\mathbb{Z})$ up to subvarieties of codimension 2, and then we shall compare ψ and φ.

Recall that the quadratic space $E_7/2E_7$ is canonically isomorphic to the subspace A'_{L_-} of q_{L_-} (Lemma 7.2). Hence Göpel subspaces bijectively correspond to totally isotropic subspaces in q_{L_-}. Consider a totally isotropic subspace A of dimension 2 in q_{L_-}. Then there exist exactly three totally isotropic subspaces I_1, I_2, I_3 in q_{L_-} containing A. Let M_1, M_2, M_3 be the corresponding Göpel subspaces. For example, M_1, M_2, M_3 are given by (4.5), (4.6), (4.4) respectively. Put $V_i = \langle I_i, \kappa \rangle$. Then it follows from Lemma 4.4 and Corollary 10.13 that $(\hat{G}_{M_1}/\hat{G}_{M_2}) = (F_{V_1}/F_{V_2})$ as divisors. Let α be a positive root in M_3 not appeared in M_1 and M_2. We can see that $\hat{G}_{M_1}/\hat{G}_{M_2}$ takes value 1 on the divisor D_α. We denote by $\mathcal{B}_a$ a component of the divisor of F_{V_3} corresponding to D_α. We may assume that $G_{M_1} - G_{M_2} = G_{M_3}$ (see Proposition 4.1, (3)). Then $\theta_{V_1} - \theta_{V_2} = \pm\theta_{V_3}$ (see Lemma 8.6) and hence $F_{V_1} - F_{V_2}$ vanishes on the divisor of F_{V_3}. This implies that F_{V_1}/F_{V_2} takes value 1 on the divisor $\mathcal{H}_a$. Hence $\hat{G}_{M_1}/\hat{G}_{M_2} = F_{V_1}/F_{V_2}$. For any pair of Göpel subspaces M'_1, M'_2, the ratio $\hat{G}_{M'_1}/\hat{G}_{M'_2}$ can be written as the product of $\hat{G}_{M_1}/\hat{G}_{M_2}$ as above type. Thus we have finished the proof of Theorem 11.1. □

11.2. **Remark.** In the above theorem, we do not consider the hyperelliptic locus. The 36 subvarieties of P_2^7 of codimension 2 in Lemma 4.3 bijectively correspond to 36 divisors $\mathcal{H}_\alpha$ (α is type $(3/2)$) (see Remark 7.3). Let $g : \tilde{P}_2^7 \to \hat{P}_2^7$ be the blow up along 36 subvarieties. Then we can see that $g^*(\tilde{G}_{V_1}/\tilde{G}_{V_2})$ has the same divisor as F_{V_1}/F_{V_2} by using Lemma 4.3. The author conjectures that the map $\hat{p}$ can be extended to the hyperelliptic locus.

REFERENCES

[AF]   D. Allcock, E. Freitag, *Cubic surfaces and Borcherds products*, Comm. Math. Helv., **77** (2002), 270–296.

[B1]   R. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math., **132** (1998), 491–562.

[B2]   R. Borcherds, *The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke Math. J., **97** (1999), 219–233.

[C]   A. Coble, *Algebraic geometry and theta functions*, Amer. Math. Soc. Coll. Publ., **10**, Providence, R.I., 1929 (3rd ed., 1969).

[DO]   I. Dolgachev, D. Ortland, *Point sets in projective spaces and theta functions*, Asterisque, **165**, 1988.

[F]   E. Freitag, *Some modular forms related to cubic surfaces*, Kyungpook Math. J., **43** (2003), 433–462.

[FS]   E. Freitag, R. Salvati-Manni, *Modular forms for the even unimodular lattice of signature* $(2, 10)$, J. Algebraic Geometry **16** (2007), 753–791.

[Kob] N. Koblitz, *Introduction to elliptic curves and modular forms*, 2nd ed., Springer-Verlag, Berlin, Heidelberg, New York 1993.

[Kon1] S. Kondō, *A complex hyperbolic structure of the moduli space of curves of genus three*, J. reine angew. Math., **525** (2000), 219–232.

[Kon2] S. Kondō, *The moduli space of Enriques surfaces and Borcherds products*, J. Algebraic Geometry, **11** (2002), 601–627.

[Kon3] S. Kondō, *The moduli space of 8 points on* $\mathbb{P}^1$ *and automorphic forms*, in "Algebraic Geometry" (ed. by J. Keum and S. Kondō), Contemporary Mathematics, **422** (2007), 89–106, AMS.

[N1]   V. V. Nikulin, *Integral symmetric bilinear forms and its applications*, Math. USSR Izv., **14** (1980), 103–167.

[N2]   V. V. Nikulin, *Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections*, J. Soviet Math., **22** (1983), 1401–1475.

[S]   B. Schoenberg, *Elliptic modular functions*, Springer-Verlag, Berlin, Heidelberg, New York 1974.

[W]   E.T. Whittaker, G.N. Watson , *A course of modern analysis*, Cambridge, Cambridge University Press 1927.

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA, 464-8602, JAPAN
E-mail address: kondo@math.nagoya-u.ac.jp